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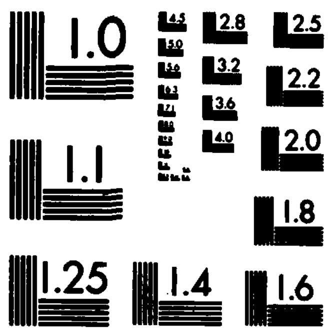

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Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

INFINITE DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATION MODELS FOR SPATIALLY DISTRIBUTED NEURONS

G. Kallianpur

and

R. Wolpert

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The membrane potential of spatially distributed neurons is modelled as a random field driven by a generalized Poisson process. Approximation to an Ornstein-Uhlenbeck type process is established in the sense of weak convergence of the induced measures in Skorokhod space.

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1. Introduction

In the neurophysiological literature, it has been well recognized that a neuron cell is spatially extended and hence that a realistic description of neuronal activity would have to take into account synaptic inputs that occur randomly in time and at different locations on the neuron's surface [(17)]. In this paper, we shall develop a stochastic model to describe the evolution of the membrane potential along the surface membrane of a nerve cell. We will regard this potential as a stochastic process or random field, indexed both by time $t>0$ and by location $x \epsilon X$ where X will represent the surface membrane of a neuron. Our work is based on and extends the work of several authors, including Wan and Tuckwell, Riccardi and Sacerdote, J. Walsh, and G. Kallianpur [21, 18, 20, 14]. In all of these earlier works the neuronal membrane X was represented by a single point [18 and 14] or by an interval $[0,b]$ of the real line. The latter model - the only spatial model treated stochastically so far as we know - has been considered by Wan and Tuckwell and analyzed probabilistically by Walsh. The deterministic background for their work is core-conductor theory and the one-dimensional cable equation which is adequate for situations which involve "longitudinal distances that are many times the cylinder diameter". Other choices of X are more realistic if one is concerned with locations "close to point sources of current and in problems where the distribution of potential in a large extracellular volume is of primary importance". [18, p. 42, also p.56].

In the present work, the form of X will be quite general; for example, it can be any smooth, compact, d-dimensional manifold. The case $d=0$ or 1 includes the work of the previous authors; $d=2$ models the surface of a neuron (useful also in the study of impulses originating at or near the soma) and $d=3$ is suitable for modelling the interior of organs such as the heart.

Before discussing the models and results it might be of interest to give a brief, though simplistic, description of the physiology of neuronal activity (see [17]).

A neuron is a cell whose principal function is to transmit information along its considerable length, which often exceeds one meter. "Information" is represented by changing amplitudes of electrical voltage potentials across the cell wall. A quiescent neuron will exhibit a resting potential of about 60 mV, the inside more negative than the outside, for reasons described below. Under certain circumstances the potential voltage in the dendritic tree will rise above a threshold point at which positive feedback causes a pulse of up to 100 mV to appear at the base of the dendritic tree; this pulse is transmitted rapidly along the body and down the axon of the cell until it reaches the so-called "pre-synaptic terminals" at the other end of the neuron. Here the pulse causes tiny vesicles filled with chemicals called "neurotransmitters" to empty out into the narrow gaps between the presynaptic terminals and the dendrites of other neurons. When these chemicals diffuse across the gap and hit the neighboring neurons' dendrites, they may cause the potential voltage in these dendrites to rise above a threshold point and initiate another pulse.

The molecules which make up neuronal cell walls (and all other living membranes) are shaped a little like lollypops with two sticks. The candy end is a phosphoric acid whose electrical charge is unequally distributed, or "polar". This allows it to mix with water molecules (which are also polar) and with ions such as sodium, potassium, and chloride. The two lollypop "sticks" are long fatty acid chains which are *not* polar; they dissolve readily in oil or fats, but not in water. These cells tend to arrange themselves so that their fatty tails meet only other fatty substances and only

their polar ends directly meet water molecules and dissolved ions; one quite stable such formation is a two-molecule-thick membrane, with all the fatty tails in the middle. This two-layer thick membrane of molecules which are half phosphoric acid and half fatty lipid are called "phospholipid bilayers".

These bilayers make extraordinary insulators and, since they are so thin, extraordinary capacitors. If for some reason a charge imbalance should arise so that one side of the membrane is more positive and the other side more negative, the insulating property of the layer would tend to maintain that imbalance and the resulting voltage potential.

In fact nearly all living membranes exhibit such a charge imbalance and resulting voltage potential, due to their selective permeability to the passage of various ions. Large protein molecules stick through the lipid bilayer; when several of these come together they can form a passage through the layer like the staves of a barrel and let one or another species of ion pass through the membrane. Different proteins have the ability to pass different species of ion selectively - say, allowing potassium to pass but not sodium - and this gives rise to the electrical potential. In the cell at rest potassium is about 30 times more plentiful inside the cells than outside, while sodium is about 10 times more plentiful outside than in. This imbalance is maintained by the ATP sodium-potassium pump, which acts like a revolving door. It actively trades Na and K to keep the potassium in and sodium out. Since the membrane is more permeable to potassium than to sodium, the positively charged potassium leaks out of the cell down its concentration gradient into the surrounding fluid, while sodium is unable to leak into the cell; the result is a net negative charge inside the cell of about 60 mV. The voltage potential itself prevents more potassium from leaking out since potassium ions are attracted by the relatively negative charge inside the cell.

Returning to the description of our model, let $\xi(t,x)$ represent the difference between the voltage potential at time t at the location $x \in X$ and the resting potential of about -60 mV. As time passes, ξ evolves due to two separate causes:

(i) Diffusion and leaks: Depending on the nature of X, the electrical properties of the cell wall may be approximated by postulating a contraction semigroup $\{T_t\}$ on $L^2(X,\Gamma)$ where Γ is a suitable σ-finite measure on X. For example, if $X = [0,b]$, core conductor theory suggests the semigroup corresponding to the diffusion equation

$$\frac{\partial \xi}{\partial t} = -\beta \xi_t + \delta \Delta \xi_t \qquad (\beta,\delta \geq 0)$$

with Neumann (or insulating) boundary conditions at both ends. In neural material like heart muscle in which electrical signals can travel more easily in some directions than in others, the Laplacian should be replaced by a more general second-order elliptic operator.

(ii) Random fluctuations: Every now and then a burst of neurotransmitter will hit some place or another on the membrane and suddenly the membrane potential will jump up or down by a random amount at a random time and location. It is believed that these random jumps are quite small and quite frequent, making it reasonable to hope that they can be modelled by a Gaussian noise process; in any case the arrivals at distant locations or in disjoint time intervals are believed to be approximately independent, justifying their modelling as a mixture of Poisson processes or as a generalized Poisson process.

Our principal concern is to prove the existence and uniqueness of solutions to stochastic differential equations (s.d.e.'s) that describe the evolution of the voltage potential ξ - a special example of such an equation is

(1.1) $$d\xi_t = (-\beta\xi_t + \delta\Delta\xi_t)dt + dX_t$$

($\beta,\delta\geq 0$) in which the "noise" or driving term, is a generalized Poisson process on $\mathbb{R}_+\times X$ or possibly a Gaussian process on $\mathbb{R}_+\times X$ - and to prove that under comparatively mild conditions a sequence of solutions to this equation with Poisson driving terms will converge in distribution (in the sense of weak convergence of the induced probability measures) to the solution with a Gaussian driving term.

These results are established in Sections 2 and 3 of this paper. In our formulation of the problem, the voltage potential ξ is viewed as a stochastic process taking values in Φ', the dual of a suitably chosen countably Hilbertian nuclear space. (Thus the stochastic differential equations we consider govern nuclear space valued processes.) It turns out that almost all the paths of the voltage potential process $\xi_{\cdot}$ lie in the Skorokhod space $D(\mathbb{R}_+ : H_{-q})$ where H_{-q} is a suitable Hilbert space.

The work of Wan and Tuckwell as well as its rigorous treatment in a more general set-up by Walsh is discussed in some detail as Example 2 of Section 4. It is perhaps appropriate here to remark briefly on the relationship of Walsh's work to ours. As in Wan and Tuckwell, Walsh takes X to be an interval $[0,b]$ and considers the potential as a stochastic process of the two parameters t, the time and x, the location. The techniques of 2-parameter martingale theory are used. The approach adopted in our paper leading to stochastic differential equations in infinite dimensional spaces (specifically Hilbert spaces or nuclear spaces) has the advantage that the theory can also be applied to more general cases in which multiparameter martingale methods are either cumbersome (e.g. if $X \subseteq \mathbb{R}^d$, $d>2$) or inapplicable (e.g. if X is a sphere or

a more general compact smooth manifold with or without a boundary). Example 3 of Section 4 and Section 5 are devoted to these applications.

Though our methods (in contrast to those of Walsh) do not permit us to study the behavior of the membrane potential process at individual points (t,x) we are able to prove stronger approximation results (see Section 4).

Stochastic differential equations in infinite dimensional spaces have been intensively studied in recent years in the context of many physical applications. They occur in the work of Dawson, Miyahara, Holley and Stroock [9, 16 and 10]. In the last named work, which is a study of infinite particle systems in statistical mechanics the authors are led to an s.d.e. driven by $S'(\mathbb{R}^d)$-valued Brownian notion. Similar equations were discussed by Itô in his Evanston lecture some years ago [12].

The models discussed in the present paper do not investigate the phenomenon of excitability, or high-speed passage of voltage pulses called "action potentials". Our models, however, are useful in describing the sub-threshold behavior of a membrane potential. Of particular interest is the probability distribution of the length of time until the base of the dendritic tree reaches a critical value θ, at which time, an action potential is generated. Information on this is given by our weak convergence result (Theorem 3.2) which implies that the first passage times for the Poisson-driven process converge in distribution to the first passage times for a generalized Ornstein-Uhlenbeck process.

In Section 6 we mention more realistic models of neuronal activity which introduce nonlinear semigroups and s.d.e.'s of Φ'-valued processes with corresponding weak convergence theorems. These questions as well as the problem of estimation of parameters of interest will be considered in a later paper.

2. Mathematical Formulation

Let $\{T_t\}_{t\geq 0}$ be a self-adjoint contraction semigroup on the Hilbert space $H = L^2(X,\mathcal{B},\Gamma)$ for some σ-finite positive measure space $(X,\mathcal{B},\Gamma)$. X is intended to represent a mass of excitable tissue (perhaps the cell membrane of a neuron), $\{T_t\}$ the evolution semigroup describing the decay of the difference ξ_t between the actual voltage potential V_t at the time $t\geq 0$ and the resting potential V_R on X. The measure Γ has no physical significance; it is chosen for convenience in order to make $\{T_t\}$ self-adjoint and to satisfy several assumptions below.

We shall require that the resolvent $R_\alpha = \int_0^\infty e^{-\alpha t}\, T_t\, dt$ be compact (for each $\alpha>0$), and even that it satisfy the assumption

A1: For some $r_1>0$ the operator $(R_\alpha)^{r_1}$ is Hilbert-Schmidt.

By the Hille-Yosida theorem $\{T_t\}$ has a negative-definite infinitesimal generator $-L$. Since $R_\alpha = (\alpha+L)^{-1}$, the assumption implies that H is separable and that L admits a complete orthonormal set $\{\phi_j\}$ of eigenvectors in H with eigenvalues $0\leq\lambda_1\leq\lambda_2\leq\ldots$ satisfying

$$\Sigma(\alpha+\lambda_j)^{-2r_1} < \infty .$$

These properties hold for any $\alpha>0$ if and only if they hold for all $\alpha>0$: we shall take $\alpha=1$ and set

$$\theta_1 = \Sigma(1+\lambda_j)^{-2r_1} \tag{2.1}$$

Denote by $<\cdot,\cdot>$ the inner product on H and let

$\Phi = \{\phi\in H\colon\ \Sigma<\phi,\phi_j>^2(1+\lambda_j)^{2r} < \infty \text{ for all } r\in\mathbb{R}\}$. For each real number r define a quadratic form $<\cdot,\cdot>_r$ and norm $\|\cdot\|_r$ on Φ by

$$<\phi,\psi>_r = \Sigma<\phi,\phi_j><\psi,\phi_j>(1+\lambda_j)^{2r} \tag{2.2}$$

$$\|\phi\|_r = (<\phi,\phi>_r)^{\frac{1}{2}}$$

and let H_r be the Hilbert-space completion of Φ in the inner product $<\cdot,\cdot>_r$. Give Φ the Fréchet topology determined by the family $\{\|\cdot\|_r\}$ of norms and let Φ' be $\cup H_r$ with the inductive limit topology.

The following are straightforward consequences of A1 and the definitions above:

i) Φ is contained in the domain of L^n for every integer n.

ii) L and T_t map Φ into itself.

iii) Finite linear combinations of $\{\phi_j\}$ are dense in Φ and in every H_r; $\{\phi_j\}$ are orthogonal in each H_r.

iv) $H_0 = H$.

v) $\Phi' \supset H_r \supset H_s \supset \Phi$ if $-\infty < r < s < \infty$; the injection of H_s into H_r is Hilbert-Schmidt if $s \geq r + r_1$.

vi) H_{-r} and H_r are in duality under the pairing $\xi[\phi] = \Sigma <\xi,\phi_j>_{-r} <\phi,\phi_j>_r$ for $\xi \in H_{-r}$, $\phi \in H_r$. The pairing is symmetric and independent of r, and $\xi[\phi] = <\xi,\phi>$ for $\xi,\phi \in H$.

vii) Φ' may be identified with the dual space (in the weak topology) to Φ.

The proof of i), for example, proceeds by showing that the sum $\Sigma <\phi,\phi_j> e^{-t\lambda_j} \phi_j$ converges in Φ for each $t \geq 0$ and each $\phi \in H$ to $T_t\phi$ and that $t^{-1}(\phi - T_t\phi) = \Sigma <\phi,\phi_j> t^{-1}(1-e^{-t\lambda_j})\phi_j$ is Cauchy in $H = H_0$ as $t \to 0$, with limit $L\phi = \Sigma <\phi,\phi_j> \lambda_j\phi_j$. Statements ii)- vi) are easy, and vii) is a consequence of the nuclear theorem (see GV, Ch 1 Thm 3).

Let μ be a σ-finite measure on $\mathbb{R} \times X$ satisfying the assumption

A2: The bilinear form

$$Q(\phi,\psi) = \int a^2\phi(x)\psi(x)\mu(dadx)$$

on $\Phi \times \Phi$ is continuous

and let $m \in \Phi'$. By the nuclear theorem there exist numbers $r_2 \in \mathbb{R}$ and $\theta_2 \in \mathbb{R}_+$ such that

$$|m[\phi]|^2 + Q(\phi,\phi) \leq \theta_2 \|\phi\|_{r_2}^2 \qquad \text{for all } \phi \in \Phi \,. \tag{2.3}$$

All processes and random variables are assumed to be defined on a fixed, but arbitrary complete probability space $(\Omega,F,\mathbb{P})$. Let N be a

regular Poisson measure on $\mathbb{R} \times X \times \mathbb{R}_+$ with mean/covariance measure $\mu(dadx)dt$ and define a Φ'-valued stochastic process $X_\bullet$ with stationary independent increments by

$$(2.4) \qquad X_t[\phi] = tm[\phi] + \int_{\mathbb{R}\times X\times(0,t]} a\phi(x)[N(dadxds) - \mu(dadx)ds] .$$

The interpretation is that $N(A\times B\times(0,t])$ should be the number of voltage pulses of sizes $a \in A \subset \mathbb{R}$ arriving at sites $x \in B \subset X$ at times $s \le t$; the probability that exactly k such pulses arrive during the indicated period is $e^{-\lambda}\lambda^k/k!$ with $\lambda = t\mu(A\times B)$. A computation will verify that

$$(2.5) \qquad \log \mathbb{E}\, e^{iX_t[\phi]} = itm[\phi] + t\int (e^{ia\phi(x)} - 1 - ia\phi(x))\mu(dadx)$$

for all $t \in \mathbb{R}_+$, $\phi \in \Phi$.

Now let $V_t = V_R + \xi_t \in \Phi'$ be the voltage potential on X at time $t \ge 0$, $V_R \in \Phi'$ the resting potential, and $\xi_t = V_t - V_R$ their difference; we will model ξ_t as the Φ'-valued solution with initial value $\xi_0 = V_0 - V_R$ to the stochastic differential equation

$$(2.6) \qquad d\xi_t = -L'\xi_t dt + dX_t .$$

Here and below L' and T_t' denote the adjoints of L and T_t when regarded as operators on Φ. In Theorem 2.1 below we construct a solution to (2.6) by evaluating the stochastic integral

$$(2.7) \qquad \xi_t = T_t'\xi_0 + \int_0^t T_{t-s}'\, dX_s .$$

Before doing so, we pause to introduce three examples; they will be developed in more detail in section 4 below.

Examples:

1. X has a single point.

   In this case $H = \mathbb{R}$ and T_t is multiplication by e^{-tL} for a constant $L \ge 0$. Obviously A1 is satisfied for any $r_1 > -\infty$.

2. X is an interval, $L = -\Delta + \beta$ (with Neumann boundary conditions).
Here we may take Lebesgue measure for Γ and verify A1 for all $r_1 > 1/4$.

3. X is the unit sphere in $\mathbb{R}^3$, $L = -\Delta$ (Laplace-Beltrami).
If we take Γ to be surface measure then A1 is satisfied for all $r_1 > \frac{1}{2}$, since then

$$\theta_1 = \Sigma(1+\lambda_j)^{-2r_1} = \sum_{\ell=0}^{\infty} (2\ell+1)(1+\ell(\ell+1))^{-2r_1} < \infty .$$

In all three examples, condition A2 is satisfied for any measure μ of the form

$$\mu(A\times B) = \sum_{k=1}^{p} 1_A(a_e^k)\nu_e^k(B) + \sum_{\ell=1}^{q} 1_A(-a_i^\ell)\nu_i^\ell(B) \tag{2.8}$$

in which $\{a_e^k\} \subset (0,\infty)$ are the possible sizes of "excitatory" (i.e. positive) pulses, $\{-a_i^\ell\}$ the sizes of "inhibitory" (i.e. negative) pulses, and $\{\nu_e^k, \nu_i^\ell\}$ are finite measures on X giving the local arrival rates. The proof hinges on the fact that in each case the orthonormal set $\{\phi_j\}$ satisfies a bound of the form $|\phi_j(x)| \le c(1+\lambda_j)^r$ uniformly in $x \in X$ and $j < \infty$; we may take $r = 0$ in examples 1 and 2, and $r = \frac{1}{4}$ in example 3.

Example 1 (with ν_e^k and ν_i^ℓ point masses of size f_e^k and f_i^ℓ, respectively on the one-point space X) appeared in [TC] and [GK] and, with slightly different notation, in [RS]. Example 2 (with $X = [0,b]$ and μ satisfying $\int_{\mathbb{R}\times X} a^2\mu(dadx) < \infty$ appears in [JW], as a generalization of the example in [WT]. In the latter example μ was of form 2.8 with each ν_e^k and ν_i^ℓ a point mass at the point x_0. Example 3 was suggested to us as a model for excitation at the soma of a neuron.* The basis for this example is the fact that synaptic inputs may occur also in the somatic region coupled with the usual assumption about the approximate spherical shape of the soma. We know of no examples other than these three which have been studied to date. We will motivate and introduce

*by Dr. T. McKenna (personal communication)

classes of new examples in Section 5.

Consider now the problem of performing the stochastic integration indicated in (2.7) for a specified random element ξ_0 of Φ'.

Since any element $\phi \in \Phi$ may be expanded into a series of the form $\phi = \Sigma < \phi, \phi_j > \phi_j$ which converges in each H_r (and therefore in Φ), we may hope to write $\xi_t[\phi] = \Sigma < \phi, \phi_j > \xi_t^j$ with $\xi_t^j = \xi_t[\phi_j]$ given by

$$(2.9) \qquad \xi_t^j = e^{-t\lambda_j} \xi_0[\phi_j] + \int_0^t e^{-(t-s)\lambda_j} dX_s[\phi_j] .$$

In order to carry out this program, introduce the notation

$$\xi_0^j = \xi_0[\phi_j] ,$$

$$m^j = m[\phi_j] ,$$

$$(2.10) \qquad X_t^j = X_t[\phi_j] ,$$

$$Y_t^j = X_t^j - tm^j ,$$

and

$$M_t^j = \int_0^t e^{s\lambda_j} dY_s^j .$$

Note that $Y_\cdot^j$ is a square-integrable martingale with covariance function $\mathbb{E} Y_s^j Y_t^k = \min(s,t)Q(\phi_i,\phi_j)$ (because $X_\cdot$ has independent increments) and $e^{s\lambda_j}$ is trivially predictable, so the martingale integral in (2.10) is well-defined and $M_\cdot^j$ is also a square-integrable martingale. We can and do take versions of $M_\cdot^j$ and $Y_\cdot^j$ (and therefore of $X_\cdot^j$) with sample paths in the Skorokhod space $D(\mathbb{R}_+:\mathbb{R})$ of right-continuous $\mathbb{R}$-valued functions on $\mathbb{R}_+$ with left-hand limits at every point of $(0,\infty)$.

Lemma 2.1

For each $T > 0$ and $q \geq r_1 + r_2$, $\mathbb{E} \sum_{j<\infty} \sup_{0 \leq t \leq T} (e^{-t\lambda_j} M_t^j)^2 (1+\lambda_j)^{-2q} \leq 16T\theta_1\theta_2$. In particular, the sum converges almost surely.

Proof:

$$e^{-t\lambda_j} M_t^j = \int_0^t e^{-(t-s)\lambda_j} dY_s^j$$

$$= Y_t^j - \int_0^t \lambda_j e^{-(t-s)\lambda_j} Y_s^j \, ds$$

so we have the bound

$$|e^{-t\lambda_j} M_t^j| \le (\sup_{0\le s\le t} |Y_s^j|)(1 + \int_0^t \lambda_j e^{-(t-s)\lambda_j} ds)$$

$$\le 2 \sup_{0\le s\le t} |Y_s^j|.$$

Doob's inequality applied to Y^j yields

$$\mathbb{E} \sup_{t\le T} (e^{-t\lambda_j} M_t^j)^2 \le 4\, \mathbb{E} \sup_{t\le T} |Y_t^j|^2$$

$$\le 16\, \mathbb{E}\, (Y_T^j)^2$$

$$= 16\; T \int a^2 \phi_j^2(x)\mu(dadx)$$

$$\le 16\; T\; \theta_2 (1+\lambda_j)^{2r_2} \quad .$$

It follows by the monotone convergence theorem that

$$\mathbb{E} \sum_j \sup_{t\le T} (e^{-t\lambda_j} M_t^j)^2 (1+\lambda_j)^{-2q}$$

$$\le 16\, T\, \theta_2 \Sigma (1+\lambda_j)^{2(r_2-q)}$$

$$\le 16\, T\, \theta_1 \theta_2 \quad ,$$

and hence the series converges almost surely. ||

Now set

(2.11) $$m_t^j = m^j \int_0^t e^{-(t-s)\lambda_j} ds$$

so that

$$\xi_t^j = e^{-t\lambda_j} \xi_0^j + e^{-t\lambda_j} M_t^j + m_t^j \, .$$

Theorem 2.1

For each random element ξ_0 of Φ' satisfying

A3: $$\mathbb{E}\,\|\xi_0\|^2_{r_3} = \theta_3 < \infty$$

for some $r_3 \in \mathbb{R}$, the series

(2.12) $$\xi_t = \Sigma\, \xi_t^j \phi_j$$

converges uniformly in $0 \le t \le T$ in the H_{-q} topology for each $T>0$ and $q \ge \max(r_1 + r_2, -r_3)$ to a process $\xi_\cdot$ whose sample paths lie in the Skorokhod space $D(\mathbb{R}_+ : H_{-q})$ of right-continuous H_{-q}-valued functions on $\mathbb{R}_+$ with left limits at each point of $(0,\infty)$. The process satisfies

$$\mathbb{E} \sup_{t \le T} \|\xi_t\|^2_{-q} \le C_T$$

and (à fortiori)

$$\mathbb{E}\,|\xi_t[\phi]|^2 \le C_T \|\phi\|^2_q \qquad (0 \le t \le T)$$

for some $C_T < \infty$.

Proof: For each $T > 0$ and $n < n' \in \mathbb{N}$ the triangle inequality yields

$$\sup_{0\le t\le T} \Big\| \sum_{n<j\le n'} \phi_j \xi_t^j \Big\|_q \le \sup_{0\le t\le T} \Big\| \sum_{n+1}^{n'} e^{-t\lambda_j} \xi_0^j \phi_j \Big\|_q \qquad \text{(Initial term)}$$

$$+ \sup_{0\le t\le T} \Big\| \sum_{n+1}^{n'} e^{-t\lambda_j} M_t^j \phi_j \Big\|_q \qquad \text{(Martingale term)}$$

$$+ \sup_{0\le t\le T} \Big\| \sum_{n+1}^{n'} m_t^j \phi_j \Big\|_q \qquad \text{(Mean term)}$$

We bound the three terms separately:

$$(\text{Initial term})^2 \le \sup_{0\le t\le T} \sum_{j>n} (e^{-t\lambda_j} \xi_0^j)^2 (1+\lambda_j)^{-2q}$$

$$\le \sum_{j>n} (\xi_0^j)^2 (1+\lambda_j)^{-2q}$$

$$\to 0 \text{ a.s. (as } n\to\infty) \text{ since } \mathbb{E}\,\|\xi_0\|^2_{-q} < \infty .$$

$$(\text{Martingale term})^2 \le \sup_{0\le t\le T} \sum_{j>n} (e^{-t\lambda_j} M_t^j)^2 (1+\lambda_j)^{-2q}$$

$$\to 0 \text{ a.s. as } n\to\infty \text{ by Lemma 2.1.}$$

$$(\text{Mean term})^2 \le \sup_{0\le t\le T} \sum_{j>n} \left(\int_0^t e^{-s\lambda_j} m^j ds\right)^2 (1+\lambda_j)^{-2q}$$

$$\le T^2 \sum_{j>n} (m[\phi_j])^2 (1+\lambda_j)^{-2q}$$

$$\to 0 \text{ as } n\to\infty \text{ since } \|m\|^2_{-q} \le \|m\|^2_{-r_2} \le \theta_2 < \infty.$$

Thus the partial sums $\sum_{j\le n} \xi_t^j \phi_j$ converge uniformly on $[0,T]$ in the H_{-q} topology. Since each partial sum lies in $D = D(\mathbb{R}_+ : H_{-q})$ this implies that the partial sums form a Cauchy sequence in the complete metric space D; let $\xi_\cdot$ denote their limit.

The estimates above and the Cauchy-Schwartz inequality together yield

$$\mathbb{E} \sup_{0\le t\le T} \|\xi_t\|^2_{-q} \le 3\mathbb{E}\left[(\text{Initial term})^2 + (\text{Martingale term})^2 + (\text{Mean term})^2\right]$$

$$\le 3[\theta_3 + 16T\theta_1\theta_2 + T^2\theta_2]$$

$$= C_T. \qquad \square$$

Theorem 2.2

For each $0\le r\le t<\infty$ the process ξ_t constructed in Theorem 2.1 satisfies the equation

$$\xi_t = T'_{t-r}\,\xi_r + \int_{(r,t]} T'_{t-s}\, dX_s \quad , \quad \text{i.e.}$$

satisfies for each $\phi \in \Phi$ the equation

$$(2.13) \qquad \xi_t[\phi] = \xi_r[T_{t-r}\phi] + \int_{(r,t]} dX_s[T_{t-s}\phi] .$$

Proof:

For any $q \geq \max(r_1 + r_2, -r_3)$ the following series converge for all $\phi \in H_q$ by Theorem 2.1:

$$\xi_t[\phi] = \Sigma\ \phi[\phi_j]\,(\xi_0^j\, e^{-t\lambda_j} + \int_{(0,t]} e^{s\lambda_j}\, dY_s^j\, e^{-t\lambda_j} + m^j \int_0^t e^{-(t-s)\lambda_j}\, ds)\ ,$$

$$\xi_r[T_{t-r}\phi] = \Sigma\ \phi[\phi_j]\ e^{-(t-r)\lambda_j}(\xi_0^j\, e^{-r\lambda_j} + \int_{(0,r]} e^{s\lambda_j}\, dY_s^j\, e^{-r\lambda_j} + m^j \int_0^r e^{-(r-s)\lambda_j} ds)$$

$$= \Sigma\ \phi[\phi_j]\,(\xi_0^j\, e^{-t\lambda_j} + \int_{(0,r]} e^{s\lambda_j}\, dY_s^j e^{-t\lambda_j} + m^j \int_0^r e^{-(t-s)\lambda_j}\, ds)\ ,$$

hence

$$\xi_t[\phi] - \xi_r[T_{t-r}\phi] = \Sigma\ \phi[\phi_j]\,(\int_{(r,t]} e^{-(t-s)\lambda_j}\, dY_s^j + \int_{(r,t]} e^{-(t-s)\lambda_j}\, m^j ds)$$

$$= \Sigma\ \phi[\phi_j]\ \int_{(r,t]} e^{-(t-s)\lambda_j}\, dX_s^j$$

$$= \int_{(r,t]} dX_s[T_{t-s}\phi]\ . \qquad \square$$

Corollary 2.1

Let $F_t = \sigma\{\xi_0, X_s:\ 0 \leq s \leq t\}$ be the smallest σ-algebra over which $\xi_0[\phi]$ and $X_s[\phi]$ are measurable for all $s \leq t$ and $\phi \in \Phi$. Then for $r \leq t$,

$$\mathbb{E}[\xi_t[\phi] | F_r] = \xi_r[T_{t-r}\phi] + \int_r^t m[T_{t-s}\phi]\ ds\ .$$

Corollary 2.2

ξ_t has the (strict) Markov property, i.e. $\mathbb{F}_r$ is conditionally independent of $\sigma\{\xi_s[\phi]:\ s \geq r,\ \phi \in \Phi\}$ given $\sigma\{\xi_r(\phi):\ \phi \in \Phi\}$.

Proof:

Let $\psi_k \in \Phi$ and $t_k \geq r$ for $1 \leq k \leq K$. Using the independent increments property of $X_\cdot$, it is possible to compute explicitly

$$\log \mathbb{E}[e^{i\Sigma\xi_{t_k}[\psi_k]} | \mathbb{F}_r] = i \sum_{k \leq K} (\xi_r[T_{t_k - r}\psi_k] + \int_r^{t_k} m[T_{t_k - s}\phi]\ ds)$$

$$+ \int_{\mathbb{R}\times X\times\mathbb{R}_+} (e^{iaF(x,s)} - 1 - iaF(x,s)\mu(dadx)ds\ ,$$

where $F(x,s) = \sum_{k\leq K} 1_{(r,t_k]}(s) T_{t_k - s}\psi_k(x)$; since this is not only $\mathbb{F}$-measurable but $\sigma\{\xi_r[\phi]: \phi \in \Phi\}$-measurable, we are done. $\square$

In the remainder of this section we state and prove results similar to Theorems 2.1 and 2.2 for stochastic differential equations of the form

$$d\zeta_t = -L'\zeta_t dt + dW_t$$

in which the Φ'-valued stochastic driving process W_t is Gaussian. The proofs are similar to (and a bit simpler than) those for the generalized Poisson process X_t already considered, so they will only be skeletal.

Several authors have considered infinite-dimensional Wiener processes and Ornstein-Uhlenbeck processes such as W_t and ζ_t in case $\Phi = S$. For example, Itô has an excellent account in his 1981 Evanston lecture, [12]. Miyahara considers similar processes in connection with a vibrating string problem [16]; Holley and Stroock, in their treatment of a problem involving infinite particle systems also introduce these processes [10]. However, since no discussion in the literature seems to include all of the estimates we will need in Section 3, we prefer to derive them here.

Now let Q denote a continuous positive-definite bilinear form on $\Phi \times \Phi$, let $m \in \Phi'$, and let $W_\bullet$ be a path-continuous Φ'-valued independent-increment stochastic process with characteristic functional

$$\mathbb{E}\, e^{iW_t[\phi]} = e^{itm[\phi] - \frac{1}{2}tQ(\phi,\phi)} ;$$

Itô [12] calls such a process a "Wiener S'-process" when $\Phi = S$. The Minlos and nuclear theorems allow us to construct such a process with continuous sample-paths lying in H_{-q} for any $q \geq r_1 + r_2$ if m and Q satisfy

A2': $$|m[\phi]|^2 + Q(\phi,\phi) \leq \theta_2 \|\phi\|^2_{r_2} \qquad \text{for all } \phi \in \Phi .$$

For random $\zeta_0 \in H_{-q}$ set

$$\zeta_0^j = \zeta_0[\phi_j] ,$$

$$m^j = m[\phi_j] ,$$

$$W_t^j = W[\phi_j] ,$$

(2.14) $$Z_t^j = W_t^j - tm^j ,$$

$$M_t^j = \int_0^t e^{s\lambda_j} dZ_s^j ,$$

$$m_t^j = \int_0^t e^{-(t-s)\lambda_j} m^j ds ,$$

and $$\zeta_t^j = e^{-t\lambda_j} \zeta_0^j + e^{-t\lambda_j} M_t^j + m_t^j .$$

Theorem 2.3

For each random element ζ_0 of Φ' satisfying

A3': $$\mathbb{E} \|\zeta_0\|_{r_3}^2 = \theta_3 < \infty \quad \text{for some } r_3 \in \mathbb{R} ,$$

the series

(2.15) $$\zeta_t = \Sigma \zeta_t^j \phi_j$$

converges uniformly in $0 \le t \le T$ in the H_{-q} topology for each $T > 0$ and $q \ge \max(r_1 + r_2, -r_3)$ to a process $\zeta_\bullet$ whose sample paths lie in the space $C(\mathbb{R}_+ : H_{-q})$ of continuous H_{-q}-valued functions on $\mathbb{R}_+$. The process satisfies

$$\mathbb{E} \sup_{0 \le t \le T} \|\zeta_t\|_{-q}^2 \le C_T$$

and (à fortiori)

$$\mathbb{E} |\zeta_t[\phi]|^2 \le C_T \|\phi\|_q^2 \qquad (0 \le t \le T)$$

for $$C_T = 3[\theta_3 + 16T\theta_1\theta_2 + T^2\theta_2] < \infty .$$

Proof:

In this Gaussian case each $Z^j_\bullet$ is an ordinary Wiener process, with diffusion rate $\mathbb{E}(Z^j_1)^2 = Q(\phi_j,\phi_j)$; it follows that $Z^j_\bullet$, $W^j_\bullet$, and $M^j_\bullet$ may all be taken to have continuous sample paths. This forces the partial sums $\sum_{j\le n} \zeta^j_t\phi_j$ to have continuous sample paths. The proof that the sequence of partial sums converges uniformly on each $[0,T]$ proceeds just as in Theorem 2.1, using exactly the same estimates and leading to the same bound C_T. $\square$

Theorem 2.4

For each $0\le r\le t<\infty$ the process ζ_t constructed in Theorem 2.3 satisfies the equation $\zeta_t = T'_{t-r}\zeta_r + \int_{(r,t]} T'_{t-s}\, dW_s$, i.e. satisfies for each $\phi\in\Phi$ the equation

$$\zeta_t[\phi] = \zeta_r[T_{t-r}\phi] + \int_r^t dW_s[T_{t-s}\phi] \tag{2.16}$$

Corollary 2.3

Let $\mathcal{F}_t = \sigma\{\zeta_0, W_s : s\le t\}$; then for $0\le r\le t<\infty$,

$$\mathbb{E}[\zeta_t[\phi]|\mathcal{F}_r] = \zeta_r[T_{t-r}\phi] + \int_r^t m[T_{t-s}\phi]\, ds \ .$$

Corollary 2.4

ζ_t has the strict Markov property.

Proof:

The theorem and its corollaries are proved in exactly the same manner as Theorem 2.2 and its corollaries. $\square$

Remark 2.1

If ζ_0 is jointly Gaussian with $\{W_\bullet\}$ then $\zeta_\bullet$ will be a Gaussian H_{-q}-valued process which is in many ways an infinite-dimensional Ornstein-Uhlenbeck (henceforth: O-U) process. If $m=0$ and ζ_0 is independent of W then for each j

$$\zeta_t^j = e^{-t\lambda_j}\zeta_0^j + \int_0^t e^{-(t-s)\lambda_j}\, dW_s^j$$

is the usual O-U process satisfying the stochastic differential equation

$$d\zeta_t^j = -\lambda_j \zeta_t^j\, dt + dW_t^j \tag{2.17}$$

with initial value ζ_0^j.

For $\lambda_j > 0$ the process ζ_t^j will be stationary if ζ_0^j is independent of $W_\cdot^j$ and has a normal distribution with mean 0, variance $Q(\phi_j, \phi_j)/2\lambda_j$; if $\lambda_j = 0$ then $\zeta_t^j = \zeta_0^j + W_t^j$ is a Wiener process starting at ζ_0^j.

Remark 2.2

The processes $\{\zeta_\cdot^j\}$ will be independent if the $\{\zeta_0^j\}$ are independent and independent of $\{W_t[\phi]: t \geq 0, \phi \in \Phi\}$ and if $\{\phi_j\}$ diagonalizes Q, i.e. if $Q(\phi_i, \phi_j) = 0$ for $i \neq j$; independence may fail in other cases. Similarly, the processes $\{\xi_\cdot^j\}$ constructed in Theorem 2.1 will be orthogonal if $Q(\phi_i, \phi_j) = \int a^2 \phi_i(x)\phi_j(x)\mu(dadx) = 0$ for $i \neq j$ (as will happen when $\Gamma(dx) = \int a^2 \mu(dadx)$, for example) and if $\mathbb{E}\,\xi_0[\phi]Y_t[\psi] = 0$ for all $t \geq 0$ and $\phi, \psi \in \Phi$.

Remark 2.3

The mean functional m in the definition of ξ_t (similarly ζ_t) plays an inessential role; in most cases it can be absorbed into ξ_0 and V_R as follows.

Assumption A1 implies that L has a finite-dimensional null-space spanned by $\{\phi_1, \ldots, \phi_n\}$ for some $n \geq 0$; let

$$m_0 = \sum_{j \leq n} m[\phi_j]\phi_j$$

be the projection of m onto that null space, and set $m_R = \sum_{j>n} (m^j/\lambda_j)\phi_j$. The series converges in H_{1-r_1} and satisfies

$$m = m_0 + Lm_R .$$

We can now write m_t^j (see (2.11) and (2.14)) in the form

$$m_t^j = m^j \int_0^t e^{-s\lambda_j}\, ds$$

$$= \begin{cases} m^j t & \text{if } j \le n \\ m^j(1-e^{-t\lambda_j})/\lambda_j & \text{if } j > n \end{cases}$$

$$= m_0[\phi_j]t + m_R[\phi_j](1 - e^{-t\lambda_j})$$

and rewrite $V_t = V_R + \xi_t$ as $\tilde{V}_R + tm_0 + \tilde{\xi}_t$ where $\tilde{V}_R = V_R + m_R$ is the new resting potential and $\tilde{\xi}_t = \xi_t - m_R - tm_0$ is the new deviation. The process $\tilde{\xi}_t$ with initial value $\tilde{\xi}_0 = V_0 - \tilde{V}_R = \xi_0 - m_R$ satisfies the equation

$$\tilde{\xi}_t[\phi] = \tilde{\xi}_0[T_t\phi] + \int_0^t dY_s[T_{t-s}\phi] \; . \tag{2.18}$$

This is just (2.7) with ξ_0 replaced by $\tilde{\xi}_0$ and X_t by the process $Y_t = X_t - tm$ which satisfies an equation like (2.4), but with $m = 0$.

The effect of m is to bring the resting potential V_R to a new equilibrium $\tilde{V}_R$. Usually L has no zero eigenvalue, so m_0 does not appear.

<u>Remark 2.4</u>

Theorems 2.2 and 2.4 suggest that the equations

$$d\xi_t = -L'\xi_t\, dt + dX_t$$

and

$$d\zeta_t = -L'\zeta_t\, dt + dW_t$$

might have stationary solutions on $-\infty<t<\infty$ satisfying (2.13) and (2.16) for all $-\infty<r<t<\infty$. When $\lambda_j > 0$ for all j this is indeed the case. After setting $m = 0$ (see Remark 2.3) and defining $X_t = Y_t$ or $W_t = Z_t$ for all $t \in \mathbb{R}$, it is easy to construct

$$\xi_t(\phi) = \int_{-\infty}^t dX_s[T_{t-s}\phi]$$

or

$$\zeta_t(\phi) = \int_{-\infty}^t dW_s[T_{t-s}\phi]$$

as in Theorems 2.1 and 2.3.

3. Weak Convergence of Solutions

Now let us fix a σ-finite measure space $(X,\mathcal{B},\Gamma)$, a self-adjoint contraction semigroup $\{T_t\}$ on $H = L^2(X,\mathcal{B},\Gamma)$, and an initial voltage distribution $\xi_0 \in \Phi'$ and consider the effect of varying the distribution of the incident noise process X_t. This distribution is uniquely determined by the mean functional $m \in \Phi'$ and the measure μ which gives the expected frequency with which impulses of various magnitudes hit X at various points.

In Theorem 2.2 we have derived the stochastic differential equation satisfied by the "electrotonic potential" ξ_t (= the difference between the voltage or membrane potential V_t and its resting value V_R). As was shown in Theorem 2.1, $\xi_\bullet$ is a process whose sample paths lie in $D(\mathbb{R}_+ : H_{-q})$. To study the continuous approximation of a sequence $\xi^n_\bullet$ of such processes it is first necessary to derive auxiliary results on the weak convergence of $D(\mathbb{R}_+ : H_{-q})$-valued processes. We begin with

<u>Lemma 3.1</u>

Let $\{P^\alpha\}$ be any family of Borel probability measures on $D = D(\mathbb{R}_+ : H)$ for some real separable Hilbert space H with norm $\|\cdot\|$. Then $\{P^\alpha\}$ is tight if and only if for each $\varepsilon > 0$ and $T > 0$ there exist a finite-dimensional subspace $F \subset H$ and positive numbers b, δ such that each P^α satisfies

(3.1) i) $P^\alpha\{h \in D: \sup\limits_{0\le t\le T} \|(I - \Pi_F)h_t\| > \varepsilon\} < \varepsilon$,

ii) $P^\alpha\{h \in D: \sup\limits_{0\le t\le T} \|\Pi_F h_t\| > b\} < \varepsilon$,

iii) $P^\alpha\{h \in D: \sup\limits_{0\le s\le \delta} \|\Pi (h_s - h_0)\| > \varepsilon\} < \varepsilon$,

$P^\alpha\{h \in D: \sup\limits_{T-\delta\le s\le T} \|\Pi_F (h_T - h_s)\| > \varepsilon\} < \varepsilon$,

and iv) $P^\alpha\{h \in D: \sup \min(\|\Pi_F (h_t - h_{t_1})\| , \|\Pi_F(h_{t_2} - h_t)\|) > \varepsilon\} < \varepsilon$.

The supremum in iv) extends over all choices of (t_1,t,t_2) satisfying $0\le t_1\le t<t_2<T$ and $t_2 - t_1 \le \delta$. In i) - iv), Π_F is the orthogonal projection in H with range F.

Proof:

By Prohorov's theorem it suffices to prove that a set $K\subset D$ has compact closure if for each $\varepsilon>0$ and $T>0$ there exist $b>0$, $\delta>0$, and $F\subset H$ as above such that every $h\in K$ satisfies

i) $\sup\limits_{0\le t\le T} \|(I-\Pi_F)h_t\| \le \varepsilon$,

ii) $\sup\limits_{0\le t\le T} \|\Pi_F h_t\| \le b$,

iii) $\sup\limits_{0\le s\le\delta} \|\Pi_F(h_s - h_0)\| \le \varepsilon, \quad \sup\limits_{T-\delta\le s\le T} \|\Pi_F(h_T - h_s)\| \le \varepsilon$,

iv) $\sup \min\{\|\Pi_F(h_t - h_{t_1})\|, \|\Pi_F(h_{t_2} - h_t)\|\} \le \varepsilon$.

The space D with its Skorokhod topology is a complete metric space under the metric

$$d(h,k) = \sup_{0<T<\infty} e^{-T}\min[1, \inf_{\lambda\in\Lambda} \max(|||\lambda|||_T, \sup_{0\le t\le T}\|h_t - k_{\lambda t}\|)]$$

where Λ is the space of strictly increasing maps from $\mathbb{R}_+$ to $\mathbb{R}_+$ and, for $T>0$, and $\lambda\in\Lambda$, $|||\lambda|||_T = \sup\limits_{0\le s<t\le T} |\log[(\lambda t - \lambda s)/(t-s)]|$. The proof is similar to that of Theorem 14.2 of Billingsley [6] . The useful property of this particular metric is that for every $T>0$ it satisfies

(3.2a) $$d(h,k) \le \max(e^{-T}, d(1_{[0,T]}h, 1_{[0,T]}k))$$

and

(3.2b) $$d(h,k) \le \max(e^{-T}, \sup_{0\le t\le T}\|h_t - k_t\|) .$$

For each $n\in\mathbb{N}$ let F_n, b_n, and δ_n be the subspace and positive numbers promised in (3.1) for $\varepsilon = e^{-n}$ and $T=n$; without loss of generality we may take $F_n\subset F_{n+1}$, $b_n\le b_{n+1}$, and $\delta_n\ge\delta_{n+1}$ for each n. This insures that

(3.3) $$\|\Pi_{F_n} h\| \le \|\Pi_{F_m} h\| \qquad \text{for } n\le m$$

for each $h \in H$.

Let $\{h_n\} \subset K$ be any sequence and define $n_{0i} = i$. By Theorem 15.2 of [6] there is a subsequence $\{n_{1i}\}$ along which $\Pi_{F_1} h_n$ converges in $D([0,1]: F)$ and hence along which $\Pi_{F_1} 1_{[0,1]} h_n$ is Cauchy in (D,d). For each $j \geq 2$ there is a further subsequence n_{ji} along which $\Pi_{F_j} 1_{[0,j]} h_n$ is Cauchy; set $k_i^j = h_{n_{ij}}$ and consider the diagonal subsequence $\{k_i^i\}$. We claim $\{k_i^i\}$ is Cauchy in D.

Fix any $\eta > 0$ and take $N \in \mathbb{N}$ large enough that $e^{-N} < \eta/3$. Let $F = F_N$ and fix $N^* \geq N$ so that for all integers $i, j \geq N^*$,

$$d(\Pi_F 1_{[0,N]} k_i^N,\ \Pi_F 1_{[0,N]} k_j^N) \leq \eta/3 .$$

From (3.2) it follows that

$$d(\Pi_F k_i^N,\ \Pi_F k_j^N) \leq \max(e^{-N},\ \eta/3) = \eta/3 . \tag{3.4}$$

Since every n_{ii} with $i \geq N$ is of the form n_{Nj} for some $j \geq i$, (3.1)-(3.4) yield for each $i, j \geq N^*$

$$d(k_i^i,\ k_j^j) \leq d(k_i^i,\ \Pi_F k_i^i) + d(\Pi_F k_i^i,\ \Pi_F k_j^j) + d(\Pi_F k_j^j,\ k_j^j) \leq \eta .$$

Since each sequence $\{h_n\} \subset K$ has a Cauchy subsequence, K has compact closure. □

Lemma 3.2

Let $\{P^\alpha\}$ be a family of probability measures on $D = D(\mathbb{R}_+ : H)$ for some real separable Hilbert space H with norm $\|\cdot\|$. A sufficient condition for $\{P^\alpha\}$ to be tight is that for every $T > 0$ and $\varepsilon > 0$ there exist a finite-dimensional subspace $F \subset H$ and positive numbers c_1, c_2, c_3 such that for every $0 \leq t_1 \leq t \leq t_2 \leq T$, each P^α satisfies

i) $\int_D \sup_{0 \leq s \leq T} \|(I - \Pi_F) h_s\|^2 \, dP^\alpha \leq \varepsilon$,

ii) $\int_D \sup_{0 \leq s \leq T} \|\Pi_F h_s\|^2 \, dP^\alpha \leq c_1$,

iii) $\int_D \|\Pi_F (h_{t_2} - h_{t_1})\|^2 \leq c_2 (t_2 - t_1)$,

iv) $\int_D \|\Pi_F(h_t - h_{t_1})\|^2 \, \|\Pi_F(h_{t_2} - h_t)\|^2 \, dP^\alpha \leq c_3(t_2 - t_1)^2$.

Proof:

Chebyshev's inequality yields i)-iii) of Lemma 3.1 while the same argument Billingsley uses in the proof of his Theorem 15.6 [6] gives iv). ||

Lemma 3.3

Let G_1 and G_2 be subsets of a topological vector-space with compact closures $\overline{G}_1$ and $\overline{G}_2$. Then $G_3 = \{g_1 + g_2 \colon g_i \in G_i,\ i=1,2\}$ has compact closure as well.

Proof:

The set $\overline{G}_1 + \overline{G}_2 = \{g_1 + g_2 \colon g_i \in \overline{G}_i,\ i=1,2\}$ is the continuous image (under addition) of the compact set $\overline{G}_1 \times \overline{G}_2 \subset V \times V$, and so is compact. This closed set contains G_3 and hence $\overline{G}_3$, which must therefore be compact. ||

Corollary 3.1

If the families $\{\mathbb{P} \circ X_\alpha^{-1}\}$ and $\{\mathbb{P} \circ Y_\alpha^{-1}\}$ of Borel measures on V induced by random elements $\{X_\alpha : \alpha \in A\}$ and $\{Y_\alpha : \alpha \in A\}$ are both tight, then the family $\{\mathbb{P} \circ Z_\alpha^{-1}\}$ induced by $Z_\alpha = X_\alpha + Y_\alpha$ is also tight (even if X_α and Y_α are not independent).

Proof:

Fix $\varepsilon > 0$ and find compact sets $G_i \subset V$ satisfying $\mathbb{P}[X_\alpha \in G_1] \geq 1 - \varepsilon/2$, $\mathbb{P}[Y_\alpha \in G_2] \geq 1 - \varepsilon/2$ for all $\alpha \in A$; then $\mathbb{P}[Z_\alpha \in G_3] \geq 1 - \varepsilon$, where (as above) $G_3 = \{g_1 + g_2 \colon g_i \in G_i,\ i=1,2\}$. ||

Theorem 3.1

Let A be any index set and let $\{m^\alpha : \alpha \in A\} \subset \Phi'$ be an equicontinuous family of linear functionals on Φ, $\{\mu^\alpha : \alpha \in A\}$ a family of measures on $\mathbb{R} \times X$ such that the bilinear forms $Q^\alpha(\phi,\psi) = \int a^2 \phi(x)\psi(x)\mu^\alpha(da\,dx)$ are equicontinuous and, for each $\phi \in \Phi$, $\int a^4 \phi^4(x)\mu^\alpha(da\,dx)$ is bounded independently of $\alpha \in A$ by some number $\theta_4(\phi) < \infty$. Let ξ_0 be a random element of Φ' satisfying

$$\mathbb{E}\,\|\xi_0\|^2_{r_3} < \infty$$

for some $r_3 > -\infty$. Then for all sufficiently large $q \in \mathbb{R}$ the family $\{P^\alpha : \alpha \in A\}$ of Borel probability measures induced on $D = D(\mathbb{R}_+ : H_{-q})$ by the processes $\{\xi^\alpha_\bullet : \alpha \in A\}$ constructed in Theorem 2.1 is tight.

Proof:

By the equicontinuity condition there exist numbers $\theta_2 > 0$ and $r_2 \in \mathbb{R}$ such that

$$|m^\alpha(\phi)|^2 + Q^\alpha(\phi,\phi) \le \theta_2 \|\phi\|^2_{r_2} \tag{3.5}$$

for all $\phi \in \Phi$ and $\alpha \in A$. Let $q \ge \max(r_1 + r_2, -r_3)$ and set $\theta_3 = \mathbb{E}\,\|\xi_0\|^2_{-q} < \infty$. It follows that assumptions A2 and A3 are satisfied for each $\alpha \in A$, so Theorem 2.1 establishes the convergence of the sum

$$\xi^\alpha_t = \sum_j \{e^{-t\lambda_j}\,\xi^j_0 + e^{-t\lambda_j}\,M^{j,\alpha}_t + m^{j,\alpha}_t\}\phi_j$$

in the D topology (and, in fact, uniformly in the H_{-q} norm on $0 \le t \le T$ for each $T > 0$). By Corollary 3.1 it suffices to show that the families $\{P^\alpha_i\}$ ($i=1,2,3$) induced by the sums

$$i = 1: \quad \xi_0(t) = \sum_j e^{-t\lambda_j}\,\xi^j_0\,\phi_j$$

$$i = 2: \quad M^\alpha_t = \sum_j e^{-t\lambda_j}\,M^{j,\alpha}_t\,\phi_j$$

$$i = 3: \quad m^\alpha_t = \sum_j m^{j,\alpha}_t\,\phi_j$$

are each tight.

$i = 1$: Since ξ_0 does not vary with α, the "family" $\{P^\alpha_1\}$ consists of a single inner-regular Borel measure on D, and hence is tight.

$i = 2$: Fix $\varepsilon > 0$ and $T > 0$. By Lemma 3.2 it is enough to find F and c_1, c_2, c_3 independent of α satisfying

i) $$\mathbb{E} \sup_{0\le s\le T} \|(I - \Pi_F)M^\alpha_s\|^2_{-q} \le \varepsilon$$

ii) $$\mathbb{E} \sup_{0\le s\le T} \|\Pi_F M_s^{\alpha}\|_{-q}^2 \le c_1$$

iii) $$\mathbb{E} \|\Pi_F (M_{t_2}^{\alpha} - M_{t_1}^{\alpha})\|_{-q}^2 \le c_2|t_2 - t_1|$$

iv) $$\mathbb{E} \|\Pi_F (M_t^{\alpha} - M_{t_1}^{\alpha})\|_{-q}^2 \|\Pi_F (M_{t_2}^{\alpha} - M_t^{\alpha})\|_{-q}^2 \le c_3|t_2 - t_1|^2 .$$

By Lemma 2.1 we can satisfy i) and ii) by setting $c_1 = 16\theta_1\theta_2$ and letting F be the space spanned by $\{\phi_1,\ldots, \phi_J\}$ for any J large enough that $16T\theta_2\sum_{j>J}(1+\lambda_j)^{-2r_1} < \varepsilon$.

Now fix any $0\le t_1\le t_2\le T$ and compute

$$\|\Pi_F (M_{t_2}^{\alpha} - M_{t_1}^{\alpha})\|_{-q}^2 = \|\sum_{j\le J} (e^{-t_2\lambda_j} M_{t_2}^{j,\alpha} - e^{-t_1\lambda_j} M_{t_1}^{j,\alpha})\phi_j\|_{-q}^2$$

$$= \sum_{j\le J} (1+\lambda_j)^{-2q}\Big(\int_{(0,t_2]} e^{-(t_2-s)\lambda_j} dY_s^{j,\alpha} - \int_{(0,t_1]} e^{-(t_1-s)\lambda_j} dY_s^{j,\alpha}\Big)^2$$

$$= \sum_{j\le J} (1+\lambda_j)^{-2q}\Big(\int_{(0,\infty)} f_j(s)dY_s^{j,\alpha}\Big)^2$$

for the function $f_j(s) = e^{-(t_2-s)\lambda_j} 1_{(0,t_2]}(s) - e^{-(t_1-s)\lambda_j} 1_{(0,t_1]}(s)$.

Note that

(3.6) $$|f_j(s)| \le \lambda_j|t_2 - t_1| 1_{(0,t_1]}(s) + 1_{(t_1,t_2]}(s) .$$

Straightforward computations with the log characteristic functional

$$\log \mathbb{E}\, e^{iY_t^{\alpha}[\phi]} = t\int(e^{ia\phi(x)} - 1 - ia\phi(x))\mu^{\alpha}(dadx)$$

show that for f,g in $L^2(\mathbb{R}_+)\cap L^4(\mathbb{R}_+)$ and ϕ,ψ in Φ,

$$\mathbb{E}\int f(t)dY_t^{\alpha}[\phi] = 0$$

(3.7) $$\mathbb{E}\Big(\int f(t)dY_t^{\alpha}[\phi]\Big)^2 = \int a^2\phi^2(x)f^2(t)\mu^{\alpha}(dadx)dt$$

$$= Q^{\alpha}(\phi,\phi)\int f^2(t)dt$$

(3.8) $$\mathbb{E}\Big(\int f(t)dY_t^{\alpha}[\phi]\Big)^2\Big(\int g(t)dY_t^{\alpha}[\psi]\Big)^2 = \Big(\int a^4\phi^2(x)\psi^2(x)\mu^{\alpha}(dadx)\int f^2(t)g^2(t)dt\Big)$$

$$+\ 2\left(\int a^2\phi(x)\psi(x)\mu^\alpha(dadx)\right)^2\left(\int f(t)g(t)dt\right)^2$$

$$+\ \left(\int a^2\phi^2(x)\mu^\alpha(dadx)\int f^2(t)dt\right)\left(\int a^2\psi^2(x)\mu^\alpha(dadx)\int g^2(t)dt\right)$$

$$= \int a^4\phi^2(x)\psi^2(x)\mu^\alpha(dadx)\int f^2(t)g^2(t)dt$$

$$+\ 2(Q^\alpha(\phi,\psi))^2\left(\int f(t)g(t)dt\right)^2$$

$$+\ Q^\alpha(\phi,\phi)Q^\alpha(\psi,\psi)\left(\int f^2dt\right)\left(\int g^2dt\right)$$

$$\le \int a^4\phi^2(x)\psi^2(x)\mu^\alpha(dadx)\int (fg)^2dt$$

$$+\ 3Q^\alpha(\phi,\phi)Q^\alpha(\psi,\psi)\left(\int f^2dt\right)\left(\int g^2dt\right) .$$

For the particular case of $\phi = \phi_j$ and $f = f_j$ we find (by (3.6) and (3.7))

$$\mathbb{E}\left(\int f_j(t)dY_t^{j,\alpha}\right)^2 = Q^\alpha(\phi_j,\phi_j)\int f_j^2(t)dt$$

$$\le Q^\alpha(\phi_j,\phi_j)[t_1\lambda_j^2|t_2-t_1|^2 + |t_2-t_1|]$$

$$\le \theta_2(1+\lambda_j)^{2r_2}[T^2\lambda_j^2+1]|t_2-t_1|$$

so iii) is satisfied with

$$c_2 = \theta_2\sum_{j\le J}(1+\lambda_j)^{-2q+2r_2}[T^2\lambda_j^2+1]$$

$$\le \theta_1\theta_2(1+T^2\lambda_J^2) .$$

If we put $\phi=\phi_i$, $\psi=\phi_j$, $f_i(s) = e^{-(t-s)\lambda_i}1_{(0,t]}(s) - e^{-(t_1-s)\lambda_i}1_{(0,t_1]}(s)$, and $g_j(s) = e^{-(t_2-s)\lambda_j}1_{(0,t_2]}(s) - e^{-(t-s)\lambda_j}1_{(0,t]}(s)$, then by (3.6) and (3.8),

$$\mathbb{E}\,\|\Pi_F(M_t^\alpha - M_{t_1}^\alpha)\|^2\,\|\Pi_F(M_{t_2}^\alpha - M_t^\alpha)\|^2$$

$$= \sum_{i,j\le J}(1+\lambda_i)^{-2q}(1+\lambda_j)^{-2q}\mathbb{E}\left(\int f_i(s)dY_s^{i,\alpha}\right)^2\left(\int g_j(s)dY_s^{j,\alpha}\right)^2$$

$$\le \sum_{i,j\le J}(1+\lambda_i)^{-2q}(1+\lambda_j)^{-2q}\left(\int a^4\phi_i^2\phi_j^2d\mu^\alpha\right)\left(\int f_i^2g_j^2dt\right)$$

$$+\ 3\sum_{i,j\le J}(1+\lambda_i)^{-2q}(1+\lambda_j)^{-2q}Q^\alpha(\phi_i,\phi_i)Q^\alpha(\phi_j,\phi_j)\left(\int f_i^2dt\right)\left(\int g_j^2dt\right)$$

$$\leq \sum_{i,j\leq J}(1+\lambda_i)^{-2q}(1+\lambda_j)^{-2q}(\theta_4(\phi_i)\theta_4(\phi_j))^{\frac{1}{2}}(\lambda_i^2\lambda_j^2 t_2^3 + \lambda_j^2 t_2)(t-t_1)(t_2-t)$$

$$+ 3\sum_{i,j\leq J}(1+\lambda_i)^{-2q}(1+\lambda_j)^{-2q}\theta_2^2(1+\lambda_i)^{r_2}(1+\lambda_j)^{r_2}(\lambda_i^2 t^2+1)(t-t_1)(\lambda_j^2 t_2^2+1)(t_2-t)$$

$$\leq c_3|t_2-t_1|^2$$

with

$$c_3 = \tfrac{1}{4}J\Big(\sum_{j\leq J}\theta_4(\phi_j)\Big)[T^3\lambda_J^4 + T\lambda_J^2](1+\lambda_J)^{4|q|} + \tfrac{3}{4}[\theta_1\theta_2(1+T^2\lambda_J^2)]^2$$

Thus i)-iv) are satisfied and $\{P_2^\alpha\}$ is tight.

$i=3$: Since m_t^α is nonrandom, each P_3^α is concentrated on a single point. It is sufficient (by Lemma 3.1) to find, for each $\varepsilon > 0$ and $T > 0$, a positive number δ such that

i) $\sup_{0\leq t\leq T}\|(I-\Pi_F)m_t^\alpha\|_{-q} \leq \varepsilon$

ii) $\sup_{0\leq t\leq T}\|\Pi_F m_t^\alpha\|_{-q} < \infty$

iii) $\sup_{0\leq s\leq\delta}\|\Pi_F[m_s^\alpha - m_0^\alpha]\|_{-q} \leq \varepsilon$

$\sup_{T-\delta\leq s\leq T}\|\Pi_F[m_T^\alpha - m_s^\alpha]\|_{-q} \leq \varepsilon$

iv) $\sup\min(\|\Pi_F[m_t^\alpha - m_{t_1}^\alpha]\|_{-q}, \|\Pi_F[m_{t_2}^\alpha - m_t^\alpha]\|) \leq \varepsilon$.

By (3.5) we have $\{m^\alpha\}\subset H_{-r_2}$ uniformly bounded by $\|m^\alpha\|_{-r_2}^2 \leq \theta_2$; pick $J\in\mathbb{N}$ large enough that $(1+\lambda_j)^{2(r_2-q)} \leq \varepsilon^2/(\theta_2 T^2)$ for $j > J$ and let $F\subset H_{-q}$ be the space generated by $\{\phi_1,\ldots,\phi_J\}$. It follows that

$$\sup_{0\leq t\leq T}\|(I-\Pi_F)m_t^\alpha\|_{-q}^2 = \sup_{0\leq t\leq T}\sum_{j>J}(1+\lambda_j)^{-2q}\Big(m^\alpha[\phi_j]\int_0^t e^{-s\lambda_j}ds\Big)^2$$

$$\leq T^2(\varepsilon^2/T^2\theta_2)\sum_{j>J}(1+\lambda_j)^{-2r_2}(m^\alpha[\phi_j])^2$$

$$\le (\varepsilon^2/\theta_2)\,\|m^\alpha\|^2_{-r_2}$$

$$\le \varepsilon^2$$

verifying i). The same estimates show that

$$\sup_{0\le t\le T} \|\Pi_F m_t^\alpha\|^2_{-q} \le T^2 \|m^\alpha\|_{-q}$$

$$\le T^2 \|m^\alpha\|_{-r_2}$$

$$\le T^2 \theta_2$$

$$< \infty$$

so ii) holds.

For iii) and iv) it is sufficient to prove that $\Pi_F m_t^\alpha$ is a uniformly continuous function of t on each finite interval, i.e. to find $\delta > 0$ so that

$$\|\Pi_F [m_t^\alpha - m_s^\alpha]\|_{-q} \le \varepsilon \quad \text{if } 0\le s\le t\le T,\ |s-t| \le \delta. \tag{3.9}$$

We compute

$$\|\Pi_F [m_t^\alpha - m_s^\alpha]\|^2_{-q} = \sum_{j\le J} (1+\lambda_j)^{-2q} (m^\alpha[\phi_j] \int_s^t e^{-u\lambda_j}\, du)^2$$

$$\le |t-s|^2 \|m^\alpha\|^2_{-q}$$

$$\le \theta_2 |t-s|^2 ,$$

so (3.9) holds for all $T > 0$ with $\delta = \varepsilon/(\theta_2)^{\frac{1}{2}}$. □

It follows from Theorem 3.1 that $\{P^\alpha\}$ has at least one cluster point, and that any cluster point is a Borel probability measure on D. We now turn to a situation in which there is a unique cluster point to which every sequence in $\{P^\alpha\}$ must converge.

Proposition 3.1

Let Y and Z be independent-increment Φ'-valued processes with log characteristic functionals

$$\log \mathbb{E}\, e^{iY_t[\phi]} = t\int (e^{ia\phi(x)} - 1 - ia\phi(x))\mu(da\,dx) \tag{3.10}$$

$$\log \mathbb{E}\, e^{iZ_t[\phi]} = -\tfrac{1}{2}tQ(\phi,\phi)$$

for some σ-finite measure μ on $\mathbb{R}\times X$ such that the positive-definite bilinear forms Q and

$$Q^{\mu}(\phi,\psi) = \int a^2\phi(x)\psi(x)\mu(da\,dx)$$

satisfy the bounds

$$Q(\phi,\phi) \le \theta\|\phi\|_r^2 \tag{3.11}$$

$$Q^{\mu}(\phi,\phi) \le \theta\|\phi\|_r^2$$

for some numbers $\theta\in\mathbb{R}_+$ and $r\in\mathbb{R}$. Extend Q and Q^{μ} to H_r by continuity.

Then there are unique continuous linear maps Y and Z from $L^2(\mathbb{R}_+ : H_r)$ to the square-integrable random variables such that

$$Y_t[\phi] = Y[1_{(0,t]}\phi], \tag{3.12}$$

$$Z_t[\phi] = Z[1_{(0,t]}\phi].$$

These maps have log characteristic functionals

$$\Psi_Y(F) = \log\mathbb{E}\, e^{iY[F]} = \int(e^{iaF_s(x)} - 1 - iaF_s(x))\mu(da\,dx)ds \tag{3.13}$$

$$\Psi_Z(F) = \log\mathbb{E}\, e^{iZ[F]} = -\tfrac{1}{2}\int Q(F_s,F_s)ds$$

which satisfy the inequality

$$|\Psi(F) - \Psi(G)| \le \tfrac{1}{2}\theta(|||F|||_r + |||G|||_r)\,|||F-G|||_r \tag{3.14}$$

for $F,G\in L^2(\mathbb{R}_+ : H_{-r})$. Here $|||F|||_r$ denotes the norm $(\int \|F_s\|_r^2 ds)^{1/2}$ of an element $F\in L^2(\mathbb{R}_+ : H_r)$.

Proof:

For functions $F\in L^2(\mathbb{R}_+ : H_r)$ of the form $F_s(x) = \sum_{j\le J} f_j(s)\phi_j(x)$ with each f_j a step function with compact support, (3.13) follows immediately from (3.10). By taking derivatives in (3.13) one sees

$$\mathbb{E}\, Y[F] = \mathbb{E}\, Z[F] = 0 \ ,$$

$$\mathbb{E}\,(Y[F]^2) = \int_{\mathbb{R}_+} Q^{\mu}(F_s, F_s)ds \le \theta |||F|||_r^2 \ ,$$

$$\mathbb{E}\,(Z[F]^2) = \int_{\mathbb{R}_+} Q(F_s, F_s)ds \le \theta |||F|||_r^2 \ .$$

Since step functions F of the form indicated are dense in $L^2(\mathbb{R}_+ : H_{-r})$, the proof will be complete (by continuity) once we verify (3.14) for step functions.

First we consider Y. If x and y are any two real numbers then

$$\begin{aligned} |(e^{ix} - 1 - ix) - (e^{iy} - 1 - iy)| &= |\textstyle\int_x^y \int_0^t i^2 e^{is}\, ds dt| \\ &\le |\textstyle\int_x^y |t| dt| \\ &\le \tfrac{1}{2}(|x| + |y|)|x - y| ; \end{aligned}$$

hence

$$\begin{aligned} |\Psi_Y(F) - \Psi_Y(G)| &\le \tfrac{1}{2}\textstyle\int a^2(|F| + |G|)|F - G| d\mu ds \\ &\le \tfrac{1}{2}(\textstyle\int a^2(|F| + |G|)^2 d\mu ds)^{\frac{1}{2}}(\int a^2|F - G|^2 d\mu ds)^{\frac{1}{2}} \\ &\le \tfrac{1}{2}\theta |||\ |F| + |G|\ |||_r\, |||F - G|||_r \\ &\le \tfrac{1}{2}\theta(|||F|||_r + |||G|||_r)\, |||F - G|||_r \ . \end{aligned}$$

For Z we use the parallelogram law:

$$\begin{aligned} |\Psi_Z(F) - \Psi_Z(G)| &\le \tfrac{1}{2}\textstyle\int |Q(F_s, F_s) - Q(G_s, G_s)| ds \\ &= \tfrac{1}{2}\textstyle\int |Q(F_s + G_s,\ F_s - G_s)| ds \\ &\le \tfrac{1}{2}(\textstyle\int Q(F_s + G_s,\ F_s + G_s) ds)^{\frac{1}{2}}(\int Q(F_s - G_s,\ F_s - G_s) ds)^{\frac{1}{2}} \\ &\le \tfrac{1}{2}\theta |||F + G|||_r\, |||F - G|||_r \\ &\le \tfrac{1}{2}\theta(|||F|||_r + |||G|||_r)\, |||F - G|||_r \ . \qquad \square \end{aligned}$$

<u>Remark 3.1</u>

For each $\psi \in \Phi'$ and $t > 0$ the random variable $M_t[\psi]$ appearing in the proof of Theorem 2.1 (resp., 2.3) is of the form $Y[F]$ (resp., $Z[F]$) with

$F_s(x) = \Sigma_j \psi[\phi_j] 1_{[0,T]}(s) e^{-(t-s)\lambda_j} \phi_j(x)$. It is easy to verify that $F \in L^2(\mathbb{R}_+ : H_r)$ if $\psi \in H_r$, since

$$|||F|||_r^2 = \int_{[0,t]} \sum_j (\psi[\phi_j]^2) e^{-2(t-s)\lambda_j} \|\phi_j\|_r^2 \, ds$$

$$\leq t \sum_j \psi[\phi_j]^2 (1+\lambda_j)^{2r}$$

$$= t\|\psi\|_r^2 .$$

Remark 3.2

For each $\psi \in \Phi$ and $t > 0$ the number $m_t[\psi]$ appearing in the proofs of Theorems 2.1 and 2.3 satisfies the inequality

$$|m_t[\psi]| = |\sum_j \psi[\phi_j] m[\phi_j] \int_{[0,t]} e^{-(t-s)\lambda_j} ds|$$

$$= |\int_{[0,t]} m[F_s] ds|$$

$$\leq |||m 1_{[0,t]}|||_{-r} |||F|||_r$$

$$= t^{\frac{1}{2}} \|m\|_{-r} |||F|||_r .$$

For each $n \in \mathbb{N}$ let X^n be an independent increment Φ'-valued process with log characteristic functional

$$\log \mathbb{E}\, e^{iX_t^n[\phi]} = itm^n[\phi] + t\int (e^{ia\phi(x)} - 1 - ia\phi(x)) \mu^n(dadx)$$

for some $m^n \in \Phi'$ and σ-finite measure μ^n on $\mathbb{R} \times X$ for which the bilinear functional

$$Q^n(\phi,\psi) = \int a^2 \phi(x)\psi(x)\mu^n(dadx)$$

is continuous on $\Phi \times \Phi$. Let W^n be an independent-increment Φ'-valued process with log characteristic functional

$$\log \mathbb{E}\, e^{iW_t[\phi]} = itm[\phi] - \tfrac{1}{2}tQ(\phi,\phi)$$

for some $m \in \Phi'$ and continuous positive bilinear form Q.

Let $\{\xi_0^n\}$ and ζ_0 all be random elements of Φ' satisfying the bounds

$$\mathbb{E}\,\|\xi_0^n\|_{r_3}^2 \le \theta_3 \tag{3.16}$$

$$\mathbb{E}\,\|\zeta_0\|_{r_3}^2 \le \theta_3$$

for some fixed $r_3 \in \mathbb{R}$ and $\theta_3 \in \mathbb{R}_+$ (independent of n), and such that each ξ_0^n is independent of $\{X_t^n\}$ and ζ_0 is independent of $\{W_t\}$. Let ξ^n and ζ be the Φ'-valued processes given by

$$\xi_t^n = T_t'\xi_0^n + \int_{(0,t]} T_{t-s}'\, dX_s^n \tag{3.17}$$

$$\zeta_t = T_t'\zeta_0 + \int_{(0,t]} T_{t-s}'\, dW_s$$

in Theorems 2.1 and 2.3, respectively. We now prove the main result of this section.

Theorem 3.2

In addition to (3.16) assume that $\{\mu^n\}$, $\{m^n\}$, and $\{\xi_0^n\}$ satisfy the following conditions:

A4: i) $\mathbb{E}\, e^{i\zeta_0[\phi]} = \lim\limits_{n\to\infty} \mathbb{E}\, e^{i\xi_0^n[\phi]}$

ii) $m[\phi] = \lim\limits_{n\to\infty} m^n[\phi]$

iii) $Q(\phi,\psi) = \lim\limits_{n\to\infty} Q^n(\phi,\psi)$

iv) $0 = \lim\limits_{n\to\infty} \int a^4\phi^4(x)\mu^n(dadx)$

v) $|m^n[\phi]|^2 + Q^n(\phi,\phi) \le \theta_2\|\phi\|_{r_2}^2$

for every ϕ and ψ in Φ, where θ_2 is independent of n. Then for every $q \ge \max(r_1 + r_2, -r_3)$ the measures $P^n = \mathbb{P}\circ(\xi^n)^{-1}$ induced on $D = D(\mathbb{R}_+ : H_{-q})$ by ξ^n converge weakly to $P = \mathbb{P}\circ(\zeta)^{-1}$.

Proof:

From ii) and iii) it follows that m and Q satisfy v) too, i.e. that

$$|m[\phi]|^2 + Q(\phi,\phi) \le \theta_2 \|\phi\|_{r_2}^2$$

for all $\phi \in \Phi$. By Lebesgue's dominated convergence theorem it follows that

$$(3.18) \qquad \|m - m^n\|_{-q}^2 = \textstyle\sum_j (m[\phi_j] - m^n[\phi_j])^2 (1+\lambda_j)^{-2q} \to 0 \quad \text{as } n \to \infty ,$$

$$(3.19) \qquad \|Q - Q^n\|_{-q\otimes -q}^2 = \textstyle\sum_{ij} (Q(\phi_i,\phi_j) - Q^n(\phi_i,\phi_j))^2 (1+\lambda_i)^{-2q}(1+\lambda_j)^{-2q} \to 0 \quad \text{as } n \to \infty .$$

The norm $\|S\|_{r\otimes s} = (\sum_{ij} S(\phi_i,\phi_j)^2 \|\phi_i\|_r^2 \|\phi_j\|_s^2)^{\frac{1}{2}}$ appearing in (3.19) is the tensor product norm $\|\cdot\|_r \otimes \|\cdot\|_s$ of the bilinear form S, when S is regarded as an element of $(\Phi\otimes\Phi)' = \bigcup_{s,r} H_r \otimes H_s$ (see [19], Def. 43.3 and the discussion following).

and satisfies $|S(\phi,\phi)| \le \|S\|_{r\otimes s} \|\phi\|_{-r} \|\psi\|_{-s}$.

Thus the hypotheses of Theorem 3.1 are satisfied, so $\{P^n\}$ is tight. To prove that P^n converges weakly to P it suffices to show that the finite-dimensional distributions of P^n converge to those of P, for then $\{P^n\}$ could have no cluster point other than P and so must converge.

Fix any $K \in \mathbb{N}$, $\{t_1,\ldots, t_K\} \subset \mathbb{R}_+$, and $\{\psi_1,\ldots, \psi_K\} \subset \Phi$. It remains only to show that

$$(3.20) \qquad \mathbb{E}\, e^{i\Sigma_k \zeta_{t_k}[\psi_k]} = \lim_{n\to\infty} \mathbb{E}\, e^{i\Sigma_k \xi^n_{t_k}[\psi_k]} .$$

Fix any $T \ge \max\{t_k\}$ and define $F \in L^2(\mathbb{R}_+ : H_q)$ by

$$F_s(x) = \textstyle\sum_k \sum_j \psi_k[\phi_j] 1_{[0,t_k]}(s) e^{-(t_k - s)\lambda_j} \phi_j(x) ;$$

note that

$$\|F_s\|_r \le (\textstyle\sum_j (\sum_k |\psi_k[\phi_j]|)^2 (1+\lambda_j)^{2r})^{\frac{1}{2}} \le \sum_k \|\psi_k\|_r$$

for any $r \in \mathbb{R}$ and $F_s = 0$ for $s \notin [0,T]$, so $F_0 \in \Phi$ and $|||F|||_r \le T^{\frac{1}{2}}\sum_k \|\psi_k\|_r$.

As in the proofs of Theorems 2.1 and 2.3 we decompose ζ and ξ^n into the three series

$$\zeta_t = (\sum_j \zeta_0[\phi_j]e^{-t\lambda_j}\phi_j) + (\sum_j \int_{[0,t]} e^{-(t-s)\lambda_j} dZ_s[\phi_j]\phi_j) + (\sum_j m[\phi_j]\int_{[0,t]} e^{-(t-s)\lambda_j} ds\phi_j),$$

$$\xi^n_t = (\sum_j \xi^n_0[\phi_j]e^{-t\lambda_j}\phi_j) + (\sum_j \int_{[0,t]} e^{-(t-s)\lambda_j} dY^n_s[\phi_j]\phi_j) + (\sum_j m^n[\phi_j]\int_{[0,t]} e^{-(t-s)\lambda_j} ds\phi_j),$$

each series converging uniformly (for $t \le T$) in the H_{-q} topology. Thus

$$\begin{aligned}\sum_k \zeta_{t_k}[\psi_k] &= \sum_k \sum_j \psi_k[\phi_j]\zeta_0[\phi_j]e^{-t_k\lambda_j} \\ &+ \sum_k \sum_j \psi_k[\phi_j]\int_{[0,t_k]} e^{-(t_k-s)\lambda_j} dZ_s[\phi_j] \\ &+ \sum_k \sum_j \psi_k[\phi_j] m[\phi_j]\int_{[0,t_k]} e^{-(t_k-s)\lambda_j} ds \\ &= \zeta_0[F_0] + Z[F] + \int_{[0,T]} m[F_s]ds\end{aligned}$$

and similarly

$$\sum_k \xi^n_{t_k}[\psi_k] = \xi^n_0[F_0] + Y^n[F] + \int_{[0,T]} m^n[F_s]ds ,$$

where Z and Y^n are the continuous linear maps from $L^2(\mathbb{R}_+ : H_q)$ into the square-integrable random variables given in Proposition 3.1.

By independence we have

$$\Psi = \log \mathbb{E}\, e^{i\sum_k \zeta_{t_k}[\psi_k]} = \Psi_{\zeta_0}(F_0) + \Psi_Z(F) + \Psi_m(F) \tag{3.21}$$

with

$$\Psi_{\zeta_0}(F_0) = \log \mathbb{E}\, e^{i\zeta_0[F_0]} ,$$

$$\begin{aligned}\Psi_Z(F) &= \log \mathbb{E}\, e^{iZ[F]} \\ &= -\tfrac{1}{2}\int_{[0,T]} Q(F_s, F_s)ds ,\end{aligned}$$

$$\Psi_m(F) = \log \mathbb{E}\, e^{i\int m[F_s]ds}$$

$$= i\int_{[0,T]} m[F_s]ds \;.$$

Similarly,

(3.22) $$\Psi^n = \log \mathbb{E}\, e^{i\Sigma_k \xi^n_{t_k}[\psi_k]} = \Psi_{\xi^n}(F_0) + \Psi_{Y^n}(F) + \Psi_{m^n}(F)$$

with

$$\Psi_{\xi_0^n}(F_0) = \log\, \mathbb{E}\, e^{i\xi_0^n[F_0]} \quad ,$$

$$\Psi_{Y^n}(F) = \int (e^{iaF_s(X)} - 1 - iaF_s(x))\mu^n(dadx)ds$$

$$\Psi_{m^n}(F) = i\int_{[0,T]} m^n[F_s]ds \;.$$

Fix $\varepsilon > 0$ and choose $J \in \mathbb{N}$ large enough that the orthogonal projection Π of H_q onto the span of $\{\phi_1, \ldots, \phi_J\}$ satisfies

$$\sum_k \|(I - \Pi)\psi_k\|_q \le \varepsilon/(12\theta_2 T^{\frac{1}{2}} |||F|||_q) \;;$$

by the triangle inequality and Remark 3.1 we have

(3.23) $$|||(I - \Pi)F|||_q \le \sum_k t_k^{\frac{1}{2}} \|(I - \Pi)\psi_k\|_q$$

$$\le \varepsilon/(12\theta_2 |||F|||_q) \,.$$

By i), iv), (3.18) and (3.19), we can find $N \in \mathbb{N}$ large enough that for each $n > N$

(3.24a) $$\left|\log \mathbb{E}\, e^{i\zeta_0[F_0]} - \log \mathbb{E}\, e^{i\xi_0^n[F_0]}\right| < \varepsilon/3 \quad ,$$

(3.24b) $$\|m - m^n\|_{-q} < \varepsilon/3T^{\frac{1}{2}} |||F|||_q \;,$$

(3.24c) $$\|Q - Q^n\|_{-q\otimes -q} < \varepsilon/12 |||F|||_q^2 \;,$$

(3.24d) $$\sum_{j\le J} \int a^4 \phi_j^4(x)\mu^n(dadx) < \varepsilon^2/4T\theta_2 |||F|||_q^2 \Big(\sum_{j\le J}\Big(\sum_{k\le K} |\psi_k[\phi_j]|\Big)^{4/3}\Big)^3 \;.$$

Introduce the temporary notation $G = \Pi F$, i.e.

$$G_s(x) = \sum_{j\le J}\sum_{k\le K} \psi_k[\phi_j]e^{-(t_k-s)\lambda_j} 1_{[0,t_k]}(s)\phi_j(x). \quad \text{Then } |||G|||_q \le |||F|||_q,$$

and

$$G_s^4(x) \le 1_{[0,T]}(s)(\textstyle\sum_j\sum_k|\psi_k[\phi_j]\phi_j|)^4$$

$$\le 1_{[0,T]}(s)(\textstyle\sum_j(\sum_k|\psi_k[\phi_j]|)^{4/3})^3\sum_j\phi_j(x)^4 ,$$

so $$\int a^4G_s^4(x)\mu^n(dadx)ds \le T(\textstyle\sum_j(\sum_k|\psi_k[\phi_j]|)^{4/3})^3 \displaystyle\sum_{j\le J}\int a^4\phi_j(x)^4\mu^n(dadx)$$

$$\le \varepsilon^2/4\theta_2|||F|||_q^2 \qquad \text{by (3.24d)}.$$

Thus

$$|\Psi_Z(G) - \Psi_{Y^n}(G)| = |\int(e^{iaG}-1-iaG+\tfrac{1}{2}a^2G^2)d\mu^n ds-\tfrac{1}{2}\int a^2G^2d\mu^n ds+\tfrac{1}{2}\int Q(G_s,G_s)ds|$$

$$\le \int|e^{iaG}-1-iaG+\tfrac{1}{2}a^2G^2|d\mu^n ds+\tfrac{1}{2}\int|Q^n(G_s,G_s)-Q(G_s,G_s)|ds$$

$$\le \frac{1}{6}\int|aG|^3d\mu^n ds + \int||Q^n - Q||_{-q\otimes -q}\,||G_s||_q^2\, ds$$

$$\le \frac{1}{6}(\int a^2G^2d\mu^n ds)^{1/2}(\int a^4G^4d\mu^n ds)^{1/2} + ||Q^n - Q||_{-q\otimes -q}\,|||G|||_q^2$$

$$\le \frac{1}{6}(\theta_2|||G|||_q^2)^{1/2}(\varepsilon^2/4\theta_2|||F|||_q^2)^{1/2} + (\varepsilon/12|||F|||_q^2)\,|||G|||_q^2$$

$$\le \varepsilon/12 + \varepsilon/12$$

$$= \varepsilon/6 .$$

By Proposition 3.1 and (3.23),

$$(3.25)\quad |\Psi_Z(F) - \Psi_{Y^n}(F)| \le |\Psi_Z(F) - \Psi_Z(G)| + |\Psi_Z(G) - \Psi_{Y^n}(G)| + |\Psi_{Y^n}(G) - \Psi_{Y^n}(F)|$$

$$\le \tfrac{1}{2}\theta_2(|||F|||_q + |||G|||_q)|||F - G|||_q + \varepsilon/6 + \tfrac{1}{2}\theta_2(|||F|||_q+|||G|||_q)|||F-G|||_q$$

$$\le \varepsilon/6 + 2\theta_2|||F|||_q|||F - \Pi F|||_q$$

$$\le \varepsilon/6 + \varepsilon/6$$

$$= \varepsilon/3.$$

By (3.21), (3.22) and (3.24b) we have

$$(3.26)\qquad |\Psi_m(F) - \Psi_{m^n}(F)| \le \int |m[F_s] - m^n[F_s]|ds$$

$$\le \int \| m - m^n\|_{-q} \|F_s\|_q ds$$

$$\le \|m - m^n\| T^{\frac{1}{2}}(\int \|F_s\|_q^2 ds)^{\frac{1}{2}}$$

$$= \|m - m^n\|_{-q} T^{\frac{1}{2}} |||F|||_q$$

$$\le \varepsilon/3 .$$

Finally by (3.21), (3.22), (3.24a), (3.25) and (3.26),

$$|\Psi - \Psi^n| \le |\Psi_{\zeta_0}(F_0) - \Psi_{\xi_0^n}(F_0)| + |\Psi_Z(F) - \Psi_{Y^n}(F)| + |\Psi_m(F) - \Psi_{m^n}(F)|$$

$$\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$

$$= \varepsilon .$$

This completes the verification of (3.20) and the proof of Theorem 3.2. ||

Remark 3.3

Condition A4 iv) can be replaced by

$$\text{A4 iv)'}\qquad \lim_{n\to\infty} \int |a|^{2+\delta} |\phi(x)|^{2+\delta} \mu^n(dadx) = 0$$

for some $\delta > 0$. Since in most applications the incident impulses are uniformly bounded by a constant $A < \infty$ and A4(iv)' and A4(iv) are equivalent in this case, we omit the necessary changes in the proof of Theorem 3.1.

4. Examples and Applications

In applications, the set X represents a neuron or some part of a neuron (such as the cell wall, soma or dendritic tree) or an assemblage of neurons and muscles such as the heart. In this section we discuss several possible mathematical models for X, the associated measure Γ and the self-adjoint contraction semigroup T_t on $L^2(X,\Gamma)$ which models the decay of voltage potentials on X in the absence of arriving excitatory and inhibitory impulses. We will also consider classes of impulse arrival-rate measures $\{\mu^n\}$ for which we can verify the hypotheses of Theorems 2.1, 3.1 and 3.2 and so construct solutions ξ^n to the equation

$$d\xi_t^n = -L'\xi_t^n\, dt + dX_t^n$$

(see Section 2) and verify that ξ^n converges weakly to a Gaussian process ξ.

First we consider in greater detail the three examples introduced in Section 2. It is convenient to state and prove the following useful lemma.

Lemma 4.1 Let T_t be a semigroup on $H = L^2(X,\Gamma)$ satisfying A1 and also

$$c_1 = \sup_x \sup_j |\phi_j(x)|\ (1+\lambda_j)^{-r} < \infty \tag{4.1}$$

for some $r < \infty$. Then any σ-finite measure μ on $\mathbb{R}\times X$ satisfying

$$c_2 = \int_{\mathbb{R}\times X} a^2\mu(dadx) < \infty \tag{4.2}$$

also satisfies assumption A2.

Proof: From (4.1) and (4.2) it follows that

$$Q(\phi,\psi) = \int a^2\phi(x)\psi(x)\mu(dadx)$$

satisfies

$$|Q(\phi_j,\phi_k)| \le \int a^2 c_1(1+\lambda_j)^r\, c_1(1+\lambda_k)^r\mu(dadx)$$

$$= c_1^2 c_2 (1+\lambda_j)^r (1+\lambda_k)^r, \qquad \text{and hence}$$

for finite sums $\phi = \sum_{j \le J} \phi[\phi_j]\phi_j$,

$$|Q(\phi,\phi)| = |\sum_{jk} \phi[\phi_j]\phi[\phi_k]\ Q(\phi_j,\phi_k)|$$

$$\le c_1^2 c_2 |\sum_j \phi[\phi_j]\ (1+\lambda_j)^r|^2$$

$$\le c_1^2 c_2 (\ \sum_j \phi[\phi_j]^2 (1+\lambda_j)^{2(r+r_1)})(\sum_j (1+\lambda_j)^{-2r_1})$$

$$= c_1^2 c_2 \theta_1 ||\phi||^2_{r+r_1} .$$

By Fatou's lemma the same bound holds for all $\phi \in \Phi$, so Q is continuous. []

Example 1. X consists of a single point.

Without loss of generality we may take $\Gamma(X) = 1$ and identify each $\phi \in \Phi$ with its value $\phi(X) \in \mathbb{R}$, in other words, we identify Φ with $\mathbb{R}$. Any contraction semigroup T_t is of the form $T_t\phi = e^{-tL}\phi$ for some $L \in \mathbb{R}_+$, and each H_s is one-dimensional with $||\phi||_s = |\phi(X)|\ (1+L)^s$. Each measure μ on $\mathbb{R}\times X$ is determined uniquely by its marginal $\mu_1(A) = \int_{A\times X}\mu(dadx)$ on the Borel sets in $\mathbb{R}$, and satisfies A2 if and only if $\int_{\mathbb{R}} a^2\mu_1(da) < \infty$. For measures of the form (2.8) (with $p \le \infty$ and $q \le \infty$), we have

(4.3) $$\mu_1(A) = \sum_{k=1}^{p} f_e^k 1_A(a_e^k) + \sum_{\ell=1}^{q} f_i^\ell\ 1_A(-a_i^\ell)$$

where $f_e^k = \nu_e^k(X)$ and $f_i^\ell = \nu_i^\ell(X)$, and A2 becomes

(4.4) $$\sum_k f_e^k (a_e^k)^2 + \sum_\ell f_i^\ell (a_i^\ell)^2 < \infty .$$

This is trivially satisfied when p and q are both finite.

A sequence μ^n of measures on $\mathbb{R}\times X$ satisfies conditions (iii)-(iv) of A4 if and only if, for some number $\sigma^2 \geq 0$, the marginals $\mu_1^n(\cdot) = \mu^n(\cdot\times X)$ satisfy

$$(4.5) \qquad \int a^2 \mu_1^n(da) \to \sigma^2$$

$$\int a^4 \mu_1^n(da) \to 0$$

as $n \to \infty$; for μ^n of form (2.8) a sufficient requirement is

$$(4.6) \qquad \sum_k f_e^{k,n}(a_e^{k,n})^2 + \sum_\ell f_i^{k,n}(a_i^{\ell,n})^2 \to \sigma^2$$

$$\max_{k,\ell}\{a_e^{k,n},\ a_i^{\ell,n}\} \to 0$$

as $n \to \infty$. In either case, for any sequence $\{m^n\} \subset \mathbb{R}$ there is a sequence of unique (in distribution) generalized Poisson processes $X_t^n = tm^n + \int_{\mathbb{R}\times(0,t]} a \times (N^n(dads) - \mu_1^n(da)ds)$ as in (2.4), satisfying (2.5) and, for each sequence ξ_0^n of square-integrable random variables, a unique (by Theorem 2.2) solution $\xi_\cdot^n$ to the stochastic initial value problem

$$d\xi_t^n = -L\xi_t^n\, dt + dX_t^n$$

$$\xi_0^n = \xi_0^n .$$

If μ^n is of the form (2.8) then $X_\cdot^n$ may be represented as

$$(4.7) \quad X_t^n = t(m^n - \bar{m}^n) + \sum_{k=1}^{p_n} a_e^{k,n}\, N_e^{k,n}(t) - \sum_{\ell=1}^{q_n} a_i^{\ell,n} N_i^{\ell,n}(t)$$

where $N_e^{k,n}$, $N_i^{\ell,n}$ $(k,\ell=1,2,\ldots)$ are independent Poisson processes with variance parameters $f_e^{k,n}$ and $f_i^{\ell,n}$ and

$$(4.8) \qquad \bar{m}^n = \sum_k f_e^{k,n} a_e^{k,n} - \sum_\ell f_i^{\ell,n} a_i^{\ell,n},$$

assuming the latter sum converges; for this it is sufficient that each

summation extend over finitely many terms.

If the sequence $\{m^n\}$ converges to a finite limit m and if $\{\xi_0^n\}$ converges in distribution, then Theorem 3.2 asserts that the sequence of Φ'-valued processes $\{\xi_\cdot^n\}$ converges weakly to an Ornstein-Uhlenback process $\zeta_\cdot$ with diffusion rate σ^2 (given by (4.6)), relaxation rate L and an additional drift coefficient m.

In particular, assuming p_n, q_n to be finite and taking $m^n = \bar{m}^n$ we obtain, in a slightly more general form, the main result of [14].

Example 2. This example includes the cases treated by Wan and Tuckwell and by Walsh [21, 20]. Let X be the interval [0,b]. If X represents the membrane of a neuron, it is natural to consider the contraction semigroup $\{T_t\}$ whose generator -L satisfies

$$L\phi = -\beta\phi + \delta\Delta\phi$$

for smooth ϕ with compact support in (0,b); $\delta \geq 0$ represents the rate of ion diffusion within the neuron, $\beta \geq 0$ the rate at which ions leak across the membrane. In the cable equation approximation to the electrical properties of neuronal membranes (see [7], for example), $\delta = (C_M(R_0+R_I))^{-1}$ cm^2/sec and $\beta = (C_M R_M)^{-1}$ sec^{-1} where R_0 and R_I are the external and internal longitudinal resistances (in ohm/cm), R_M the membrane resistance per unit length (in ohm-cm) and C_M the membrane capacitance per unit length (in F/cm). If we select Lebesgue measure for Γ and impose Neumann boundary conditions at 0 and b (i.e., seal and insulate the ends) then L and $\{T_t\}$ are each uniquely determined and self-adjoint on $H = L^2(X,\Gamma)$. L has eigenfunctions $\{\phi_j\}$ with associated eigenvalues $\{\lambda_j\}$ given by

$$\phi_j(x) = C_j \cos(\pi j x/b) \qquad j=0,1,2\ldots$$

$$\lambda_j = \beta + \delta(\pi j/b)^2.$$

If we set $C_0 = b^{-\frac{1}{2}}$ and $C_j = 2^{\frac{1}{2}}b^{-\frac{1}{2}}$ for $j \geq 1$ then $\{\phi_j\}$ is a complete orthonormal set in H. Assumption A1 is satisfied for any $r_1 > \frac{1}{4}$; for example, $r_1=1$ yields

$$\theta_1 = \sum_j (1+\lambda_j)^{-1} = \tfrac{1}{2} b\,((\beta+1)\delta)^{-\frac{1}{2}} \coth(b(\beta+1)^{\frac{1}{2}}\delta^{-\frac{1}{2}}) + \tfrac{1}{2}(\beta+1)^{-1}.$$

By Lemma 4.1 (with r=0), A2 is satisfied by any measure μ on $\mathbb{R}\times[0,b]$ satisfying (4.2); for sufficiently smooth m, we may take $r_2 = r_1$ in (2.3). With the choice of X, H and T_t as above consider the Wan-Tuckwell model which assumes white noise current injection at a single point $x_0 \in [0,b]$, [21]. The impulse arrival-rate measures μ^n are taken to be of the form

$$\mu^n(A\times B) = \Big[\sum_{k=1}^{p} f_e^{k,n}\, 1_A(a_e^{k,n}) + \sum_{\ell=1}^{q} f_i^{\ell,n} 1_A(-a_i^{\ell,n})\Big]\cdot 1_B(x_0). \tag{4.9}$$

(4.9) is a special case of (2.8). Here and throughout this example we will assume p, q < ∞ and independent of n. We have

$$Q^n(\phi,\psi) = \phi(x_0)\psi(x_0)\ \sigma_n^2$$

where ϕ and $\psi \in \Phi$ and

$$\sigma_n^2 = \sum_{k=1}^{p} f_e^{k,n}(a_e^{k,n})^2 + \sum_{\ell=1}^{q} f_i^{\ell,n}(a_i^{\ell,n})^2. \tag{4.10}$$

Under the assumption

$$\lim_{n\to\infty} \sigma_n^2 = \sigma^2 \qquad (0 \leq \sigma^2 < \infty), \tag{4.11a}$$

and

$$\lim_{n\to\infty}\ \max_{k,\ell}\{a_e^{k,n},\ a_i^{\ell,n}\} = 0, \tag{4.11b}$$

the assumptions (A4) (iii)-(v) are easily seen to be satisfied with

(4.12) $$Q(\phi,\psi) = \phi(x_0)\psi(x_0)\ \sigma^2.$$

The white noise processes $X^n_\cdot$ then have the representation

(4.13) $$X^n_t(\phi) = t[m^n(\phi) - \bar{m}^n(\phi)] + \sum_{k=1}^{p} a_e^{k,n} \int_{X\times(0,t]} \phi(x) N_e^{k,n}(ds,dx) - \sum_{\ell=1}^{q} a_i^{\ell,n} \int_{X\times(0,t]} \phi(x)\ N_i^{\ell,n}(ds,dx),$$

where $N_e^{k,n}$, $N_i^{\ell,n}$ are independent Poisson measures with variance measures respectively given by $f_e^{k,n}\cdot\nu(dx)$ and $f_i^{\ell,n}\ \nu(dx)$ with $\nu(B) = 1_B(x_0)$. In (4.13) $\bar{m}^n$ is given by

(4.14) $$\bar{m}^n(\phi) = \gamma_n \phi(x_0)$$

with

$$\gamma_n = \sum_{k=1}^{p} f_e^{k,n}\ a_e^{k,n} - \sum_{\ell=1}^{q} f_i^{\ell,n}\ a_i^{\ell,n}\ .$$

Consistent with the Wan-Tuckwell model, choose $m^n(\cdot) = \bar{m}^n(\cdot)$ and assume that

(4.15) $$\lim_{n\to\infty} \gamma_n = \gamma, \qquad \gamma<\infty \ \text{ exists.}$$

Furthermore, assume ξ_0^n to satisfy A4 (i). Under the assumptions (4.9)-(4.11) and (4.15) it follows from Theorem 3.2 that the processes $\xi^n_\cdot$ satisfying

$$d\xi_t^n = -L'\xi_t^n\ dt + dX_t^n, \qquad (t>0)$$

with ξ_0^n as above, converge weakly to the Φ'-valued Gaussian process $\zeta_\cdot$ given by

(4.16) $$\zeta_t = \sum \phi_j \zeta_t^j\ .$$

The real-valued process $\zeta^j_\cdot$ satisfy the stochastic differential equation

$$(4.17)\qquad d\zeta^j_t = [-\lambda_j\zeta^j_t + \gamma\phi_j(x_0)]dt + \sigma|\phi_j(x_0)|dw^j_t$$

with initial value ζ^j_0. Each w^j is a standard, real-valued Wiener process. Equation (4.17) is equation (25) obtained by Wan and Tuckwell in [21]. We note, however, that in [21] p and q are both taken to be equal to 1. As a consequence of Theorem 2.3 it also follows that the series in (4.16) converges uniformly in $0 \le t \le T$ in the H_{-q} topology where $q > \max(\frac{1}{2},-r_3)$.

In their paper Wan and Tuckwell briefly comment on an extension of their model in which the impulses can arrive at different points of X, i.e., instead of all the impulses arriving at a single point x_0, there are distinct points x^k_e $(k=1,\ldots,p)$ at which excitatory impulses arrive and distinct points x^ℓ_i $(\ell=1,\ldots,q)$ which receive only inhibitory impulses. The authors state that the locations of the excitatory and inhibitory synapses "do not differ by very much".

In place of (4.9) we now must take

$$(4.9)'\qquad \mu^n(A\times B) = \sum_{k=1}^{p} f^{k,n}_e\, 1_A(a^{k,n}_e)1_B(x^k_e) + \sum_{\ell=1}^{q} f^{\ell,n}_i\, 1_A(-a^{\ell,n}_i)I_B(x^\ell_i).$$

Then

$$Q^n(\phi,\psi) = \sum_{k=1}^{p} f^{k,n}_e(a^{k,n}_e)^2\phi(x^k_e)\psi(x^k_e) + \sum_{\ell=1}^{q} f^{\ell,n}_i(a^{\ell,n}_i)^2\phi(x^\ell_i)\psi(x^\ell_i),$$

and the corresponding $\bar{m}^n(\phi)$ is of the form

$$(4.18)\qquad \bar{m}^n(\phi) = \sum_{k=1}^{p} f^{k,n}_e\, a^{k,n}_e\, \phi(x^k_e) - \sum_{\ell=1}^{q} f^{\ell,n}_i\, a^{\ell,n}_i\, \phi(x^\ell_i).$$

As before, we will take $m^n = \bar{m}^n$ in the choice of the noise process X^n. The assumption

$$Q^n(\phi,\psi) \to Q(\phi,\psi) \quad \text{for all } \phi,\psi \text{ in } \Phi$$

leads to the existence of the finite limits (assumed positive)

(4.19) $\quad \alpha_e^k = \lim_{n\to\infty} f_e^{k,n}(a_e^{k,n})^2, \quad \alpha_i^\ell = \lim_{n\to\infty} f_i^{\ell,n}(a_i^{\ell,n})^2$ for $k=1,\ldots,p$

and $\ell=1,\ldots,q$. We then have

$$Q(\phi,\psi) = \sum_{k=1}^{p} \alpha_e^k\, \phi(x_e^k)\psi(x_e^k) + \sum_{\ell=1}^{q} \alpha_i^\ell\, \phi(x_i^\ell)\psi(x_i^\ell).$$

This assumption and the additional condition (4.11b) verifies (A4) (iii)-(v). (A4) (i) takes the form

(4.20) $\quad \lim_{n\to\infty} \overline{m}^n(\phi) = \overline{m}(\phi)$, finite for every $\phi \in \Phi$.

However, conditions (4.19), (4.11b) and (4.20) are incompatible unless

(4.21) $\quad p = q$ and $x_e^k = x_i^k \quad (k=1,\ldots,p)$.

(4.21) can be easily shown by using the fact that C^∞ functions with compact support belong to Φ. Choose such a function ϕ with $\phi(x_e^k) = 1$, $\phi(x_e^m) = \phi(x_i^\ell) = 0$ for all $\ell = 1,\ldots,q$ and $m\neq k$, where x_e^k is a point not belonging to the set $\{x_i^\ell, \ell=1,\ldots,q\}$. Then (4.20) implies $f_e^{k,n}\, a_e^{k,n} \to$ a finite limit as $n \to \infty$ which is impossible in view of (4.19) and (4.11b). Thus all the points x_e^k belong to the set $\{x_i^\ell, \ell=1,\ldots,q\}$. Similarly, all the x_i^ℓ belong to the set $\{x_e^k, k=1,\ldots,p\}$ and thus the two sets are the same. Renumbering the points $\{x_i^\ell\}$ if necessary, we have (4.21). Writing $x_e^k = x_i^k = x_k$ $(k=1,\ldots,p)$ we have

$$Q(\phi,\phi) = \sum_{k=1}^{p} (\alpha_e^k + \alpha_i^k)\, \phi^2(x_k)$$

and

$$\overline{m}(\phi) = \sum_{k=1}^{p} \gamma^k\, \phi(x_k)$$

for all $\phi \in \Phi$ where $\gamma^k = \lim_{n\to\infty}[f_e^{k,n}\ a_e^{k,n} - f_i^{k,n}\ a_i^{k,n}]$. Note that the argument just given holds for any X and Φ which contains functions that distinguish points of X. The stochastic differential equation for the processes ζ^j corresponding to (4.17) now takes the form

$$(4.22) \qquad d\zeta_t^j = [-\lambda_j\zeta_t^j + \overline{m}(\phi_j)]dt + \sqrt{Q(\phi_j,\phi_j)}\ dw_t^j$$

with initial value ζ_0^j and w^j a standard Wiener process. The Φ'-valued process ζ is given by (4.16) with ζ^j satisfying (4.22). The processes ξ^n satisfy an equattion very similar to that obtained in the simpler Wan-Tuckwell model with white noise current injection at a single point. Equation (4.22) differs from (4.17) only in the constants $\overline{m}(\phi_j)$ and $\sqrt{Q(\phi_j,\phi_j)}$.

Thus if $\xi_\bullet$ is a solution to the initial value problem

$$d\xi_t = -L'\xi_t dt + dX_t$$

$$\xi_0 = \xi_0$$

in which $X_\bullet$ is a generalized Poisson process with only very small jumps, then $\xi_\bullet$ is close in distribution to the solution $\zeta_\bullet$ to the initial value problem

$$d\zeta_t = -L'\zeta_t dt + dW_t$$

$$\zeta_0 = \zeta_0$$

If $W_\bullet$ is a Gaussian Φ'-valued process with the same mean and covariance as $X_\bullet$ and if ζ_0 is a Gaussian Φ'-valued random variable whose distribution is close to that of ξ_0. This supports the use of Gaussian methods to study the approximate distribution of the process $\xi_\bullet$, including first-passage times and related functionals (e.g. in [21] and [18]).

In [20] Walsh considers a model in which X, Γ and $\{T_t\}$ are the same as in the Wan-Tuckwell example discussed above but the choice of μ is more general, i.e., μ is determined by a finite measure ν on X and a Markov kernel K via the relation

$$\mu(A\times B) = \int_B K(x,A)\ \nu(dx), \qquad A \subset \mathbb{R},\ B \subset X.$$

The kernel $K(x,da)$ is the regular conditional probability distribution of the size $a \in \mathbb{R}$ of impulses arriving at the site $x \in X$, while the measure $\nu(B)$ is the overall arrival rate $\mu(\mathbb{R}\times B)$ of impulses of all sizes at points $x \in B \subset X$. The requirement made in [20] that $\int[\int a^2 K(x,da) + (\int a\ K(x,da))^2]\ \nu(dx) < \infty$ entails that 4.2 is satisfied so that A2 holds and Theorems 2.1 and 2.2 apply. Incidentally, Walsh represents T_t through its integral kernel $G(x,y;t) = \sum_j e^{-t\lambda_j}\ \phi_j(x)\ \phi_j(y)$. The series converges uniformly in x and y by A1. Walsh defines a two-parameter stochastic process $V(t,x)$ which is an integral kernel for our ξ_t:

$$\xi_t[\phi] = \int \phi(x)\ V(t,x)\ \Gamma(dx)$$

and our notation is related to his as follows:

$$\xi_0[\phi] = \int_X \phi(x)\ v_0(x)\Gamma(dx),$$

$$\begin{aligned}
\xi_t[\phi] &= \int_X \phi(y)\ V(t,y)\Gamma(dy) \\
&= \int_X \phi(y) \int_X v_0(x)\ G(x,y;t)\Gamma(dx)\Gamma(dy) \\
&\quad + \int_X \phi(y)\ [\int_X \int_0^t G(x,y;t-s)\ F(dsdx)]\Gamma(dy) \\
&= \int_X [\int G(x,y;t)\phi(y)dy]\ v_0(x)\Gamma(dx) \\
&\quad + \int_0^t \int_X [\int G(x,y;t-s)\phi(y)\Gamma(dy)]\ F(dsdx)
\end{aligned}$$

$$= \int_X [T_t\phi(x)]\ v_0(x)\Gamma(dx)$$

$$+ \int_0^t \int_{\mathbb{R}\times X} [T_{t-s}\phi(x)]\ aN(dadxds)$$

$$= \xi_0[T_t\phi] + \int_{(0,t]} dX_s[T_{t-s}\phi].$$

It is shown in Theorem 3.3 of [20] that, under certain conditions (3.1 of [20]), almost all of the finite-dimensional distributions of a sequence of such processes V^n converge to those of a Gaussian process V^∞ which is an integral kernel for our ζ_t. Walsh's condition (3.1a) implies our A4 (iii) while his (3.1b) is equivalent to our A4 (iv)'. It should be noted, though, that our approach to the problem is different from Walsh's. Our results are not concerned with the convergence in distribution of $V^n(t,x)$ for fixed t and x. On the other hand, Theorem 3.2 of our paper stengthens Theorem 3.3 of [20] by proving the convergence in distribution of the Φ'-valued processes $\xi^n_\cdot$ in the sense that the probability measures in $D(\mathbb{R}_+:\Phi')$ induced by the $\xi^n_\cdot$'s converge weakly to that induced by $\zeta_\cdot$. The advantage of establishing weak convergence of the Φ'-valued processes is that it yields convergence in distribution of continuous functionals of the paths, such as the first passage times, $\tau(\phi) = \inf\{t>0: \xi_t(\phi)>\theta\}$, $= \infty$ if $\{\ldots\}$ is the null event, ($\phi \in \Phi$ or H_q).

As already noted earlier, our approach to modelling the biological phenomena of interest as Φ'-valued processes is very general as is clear from the assumptions made in Theorem 3.1 on X, Γ and the semigroup $\{T_t\}$, and as will be further illustrated in the next example and Section 5. In these examples, X will be a sphere in $\mathbb{R}^3$ or a smooth, compact manifold with or without boundary. Such problems lie outside the scope of multi-parameter martingale theory. Even

when $X \subset \mathbb{R}^d$ $(d \geq 1)$, the semigroup $\{T_t\}$ cannot, in general, be represented by a kernel given by a uniformly convergent series, the case treated by Walsh. When the series makes sense as a distribution, one is naturally led to a generalized stochastic process of the kind we have considered in this paper.

<u>Example 3</u>. X consists of a sphere.

Let $X = S^2$, the unit sphere in $\mathbb{R}^3$, with Lebesgue surface measure Γ. If Δ_B denotes the spherical Laplace-Beltrami operator

$$\Delta_B \phi = (\sin\theta)^{-1}\left[\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial\phi}{\partial\theta} + \frac{\partial}{\partial\eta}(\sin\theta)^{-1}\frac{\partial\phi}{\partial\eta}\right]$$

(where θ,η are the Euler angles on S^2), then $L = -\beta + \delta\Delta_B$ is self-adjoint on $H = L^2(X,\Gamma)$ for any real numbers β, δ. If β and δ are positive then $-L$ generates a contraction semigroup $\{T_t\}$ as before. This time the eigenfunctions are the spherical harmonics $Y_{\ell m}$ $(\ell=0,1,\dots;\ m=-\ell,\dots,\ell)$ with eigenvalues $\lambda = \beta+\delta\ell(\ell+1)$ for L, $e^{-t\lambda}$ for T_t; we set

$$\phi_j = Y_{\ell m} \qquad j=m+\ell(\ell+1) \quad (\text{i.e. } \ell=[[j^{1/2}]],\ m=j-\ell^2-\ell)$$

$$\lambda_j = \beta+\delta\ [[j^{1/2}]]\ ([[j^{1/2}]]+1)$$

Assumption A1 is satisfied for any $r_1 > \frac{1}{2}$, since

$$\begin{aligned}\theta_1 = \textstyle\sum_j (1+\lambda_j)^{-2r_1} &= \textstyle\sum_\ell (2\ell+1)(1+\beta+\delta(\ell^2+\ell))^{-2r_1} \\ &< 4\beta^{-2r_1} + \delta^{-2r_1}/(r_1-\tfrac{1}{2}) \\ &< \infty.\end{aligned}$$

The spherical harmonics obey the bound

$$\sup_x |Y_{\ell m}(x)|^2 = (2\ell+1)/4\pi$$

so $\{\phi_j\}$ satisfies

$$\sup_j \sup_x |\phi_j(x)| \, (1+\lambda_j)^{-\frac{1}{4}} = (4\pi^2 \min(\delta,4\beta))^{-\frac{1}{4}}$$
$$< \infty.$$

By Lemma 4.1, any μ on $\mathbb{R}\times X$ satisfying (4.2) also satisfies A2. For further details the reader may consult [5].

5. Applications Where X is a Compact Riemannian Manifold

Often the set X is intended to be the cell wall or surface membrane of a neuron; as such it ought to admit a mathematical representation as a compact Riemannian manifold of two dimensions. In voltage-clamp experiments the cell wall is cut and the membrane potential voltage at the cut is held at some prescribed value by the experimental apparatus; in this case X may be regarded as a manifold with boundary.

The following theorem gives conditions on a manifold-with-boundary X, a measure Γ on X, and a semigroup $\{T_t\}$ on $L^2(X,\Gamma)$ which guarantee that assumptions A1) and A2) will be satisfied for any measure μ satisfying (4.2), i.e. for any noise process X_t admitting mean and covariance measures. See Hörmander ([11], ch. X) for details about elliptic boundary systems on manifolds.

Theorem 5.1

Let X be a smooth d-dimensional compact Riemannian manifold with smooth (possibly empty) boundary ∂X and Riemannian volume element $d\Gamma$. Let L be a positive self-adjoint operator on a domain $\mathcal{D} \subset H = L^2(X,\Gamma)$ satisfying

i) $C_c^\infty(X) \subset \mathcal{D}$

ii) The restriction L_0 of L to $C_c^\infty(X)$ is a uniformly strongly elliptic differential operator of order $2m > 0$ with smooth coefficients.

iii) $\mathcal{D} \subset W_{2m}(X)$, the Hilbert space of those elements in H with $2m$ weak derivatives in H.

Then L admits a complete orthonormal set $\{\phi_j\}$ of eigenfunctions in H with eigenvalues $\{\lambda_j\}$ satisfying

i) $L\phi_j = \lambda_j \phi_j$

ii) $\phi_j \in C^\infty(X)$

iii) $\sum_j (1+\lambda_j)^{-2r_1} < \infty$ for all $r_1 > d/4m$

iv) $\sup_j \sup_x |\phi_j(x)|(1+\lambda_j)^{-r} < \infty$ for all $r > d/4m$.

Corollary 5.1

The contraction semigroup T_t generated by $-L$ satisfies A1. Any measure μ on $\mathbb{R}_+ \times X$ satisfying (4.2) must also satisfy A2.

Proof of Theorem 5.1

In case X is an open set in $\mathbb{R}^d$, Theorem 14.6 of Agmon [1] gives i) and iii); ii) follows from elliptic regularity (e.g. [1] Theorem 9.3) and iv) from the Sobolev imbedding theorem (e.g. [1] Theorem 3.9). Completeness of $\{\phi_j\}$ follows from [2], Theorem 3.4.

The case of a general Riemannian manifold can be treated by extending Agmon's proof to manifolds (using the Schauder estimates of [3]) or by employing a partition-of-unity argument as in Hörmander ([11] Chapter X).

Remark 5.1

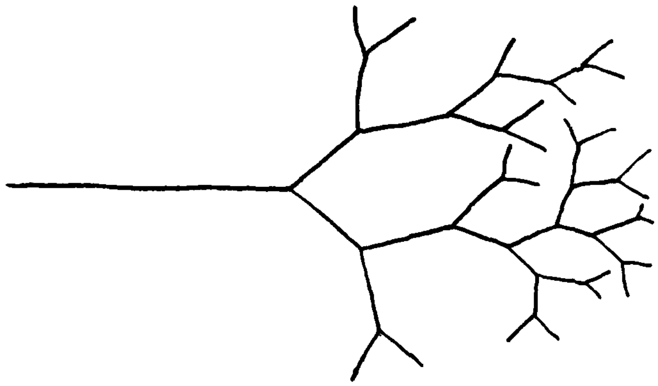
Frequently in applications L_0 is given as part of an elliptic boundary system $L = \{L_0; \ell_1, \ldots, \ell_m\}$ (see [11], p. 254) in which $\ell_1, \ldots, \ell_m$ are smooth differential operators of order < 2m on $C^\infty(\partial X)$. Theorem 5.1 will apply if it can be shown that L_0 is essentially self-adjoint on the domain $\mathcal{D}_0 = \{\phi \in C^\infty(X) : \phi$ has a continuous extension $\bar{\phi}$ to $\bar{X} = X \cup \partial X$, the restriction of $\bar{\phi}$ to ∂X is C^∞, and $\ell_j \bar{\phi} \equiv 0$ on ∂X for $1 \le j \le m\}$.

The second-order elliptic operators with Dirichlet or Neumann boundary conditions can be treated in this way.

Examples 1, 2 and 3 of Section 4 are special cases of this, with $d = 0$, $d = 1$, and $d = 2$ respectively. A more realistic example would appear to be that in which X is the boundary of a 3-dimensional solid, so $\partial X = \emptyset$, and $L = -\Delta_B + \beta(x)$ is the sum of the Laplace-Beltrami operator and a smooth positive function representing the trans-membrane conductance on a (not necessarily spherical) 2-manifold X.

Example 4.

Before simplifying to $X = [0,b]$ Wan and Tuckwell discuss a one-dimensional finitely branching tree as a model for X:



Although this is not quite a manifold (it is not locally homeomorphic to $\mathbb{R}$ at the nodes) it nonetheless can be handled by our methods as follows.

Represent X as the disjoint union of N compact intervals $[0,b_1],\ldots,[0,b_N]$ and each function ϕ on X as a family $\phi_1,\ldots,\phi_N$ of functions on the intervals satisfying certain boundary conditions. If J_i represents the (possibly empty) set of indices for the segments emerging from the node at the end of the ith segment, the appropriate boundary conditions are

1) $\phi_i(b_i) = \phi_j(0)$ for each $j \in J_i$, $1 \le i \le N$,

2) $\phi_i'(b_i) = \sum_{j \in J_i} \phi_j'(0)$ for each $1 \le i \le N$,

3) $\phi_1'(0) = 0$.

The first of these guarantees that ϕ will be continuous when regarded as a function on the tree, while the second is an expression of Kirchhoff's Law forbidding any leak of current at the nodes. The third equation imposes Neumann

(or "insulating") boundary conditions at the base of the tree; it or any of the N conditions in 2) could be replaced by a Dirichlet (or "grounded") condition. It is easy to integrate by parts and show that $\Delta\phi_i = \phi_i''$ is self-adjoint on the domain $\mathcal{D}$ of all C^1 functions on X satisfying the boundary conditions above and possessing an absolutely continuous derivative and a square-integrable second derivative with respect to Lebesgue measure Γ.

Theorem 5.1 still doesn't quite apply since the boundary operators are non-local (on the disconnected set X), but its conclusion still holds true. In fact, $-\Delta + \beta$ has a uniformly bounded complete orthonormal set of C^∞ eigenvectors and hence one can take $r = 0$ instead of just $r \geq 1$ in conclusion iv).

6. Nonlinear Problems and Directions for Future Work

The stochastic behavior of the voltage potential investigated in the previous sections has the form of a differential equation

$$(6.1) \qquad d\xi_t = -L'\xi_t dt + dX_t$$

where $L' : \Phi' \to \Phi'$ is the adjoint of L. In our model L' is a linear operator and hence $\{T_t'\}$ appearing in (3.17) is a linear evolution semigroup. This is a decidedly unrealistic feature, for physically it amounts to the assumption that the electrical properties of the membrane are unaffected by changes in the potential voltage across the membrane. Our model fails to take into account experimentally observed features of certain neurons, for instance the following:

(a) As the cell membrane is progressively depolarized, the postsynaptic potentials (inhibitory as well as excitatory) eventually become reversed in sign. The reversal potential would introduce one type of nonlinearity in the model (see [21])

(b) At least in some cells (such as sympathetic ganglion cells, see [4], p. 135) the sizes of the postsynaptic potential impulses are dependent on the state of depolarization of the membrane potential.

It would thus seem that a model which better describes the physiological process would let ξ_t satisfy an equation such as

$$(6.2) \qquad d\xi_t = a_t' dt + b_t' dX_t$$

in which a' is a Φ'-valued process adapted to $\{X_t\}$ (most likely a nonrandom, nonlinear Φ'-valued function of ξ_t) and b_t' is an adapted process taking values in a space of functions from Φ' to Φ'.

We can replace $X_\cdot$ by $W_\cdot$ and $\xi_\cdot$ by $\zeta_\cdot$ in (6.1) and (6.2) and seek to approximate the behavior of the process ξ_t by that of a nuclear space-valued diffusion ζ_t satisfying

$$(6.3) \qquad d\zeta_t = a_t' dt + b_t' dW_t ,$$

but the techniques used in the present paper do not suffice to prove the existence and uniqueness of solutions to (6.2) and (6.3) or to prove the distributional convergence of a sequence of solutions to (6.2) to the corresponding solution of (6.3) when a' and b' are nonlinear.

Even the use of a semigroup (linear or not) $\{T_t\}$ to model the evolution of the membrane potential in the absence of incident impulses entails an implicit assumption about the physical system which is unrealistic, namely that the state of the system is unambiguously specified by giving only the membrane potential itself at every point of X. It is known [13] that the local membrane behavior depends critically on the concentration gradients of sodium, potassium, chloride, and calcium ions, and that active transport of these ions as well as diffusion play important roles. Recent experiments revealing the stochastic behavior of ion-specific gates through cell membranes offer new opportunities for more elaborate stochastic modelling of this important system.

Finally, the stochastic differential equation model with which this paper deals and its extensions briefly indicated above have applications to other areas of biology, e.g. to problems of emigration of biological populations. We hope to investigate such problems in future papers.

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